RATIONAL CURVES ON COMPACT KÄHLER MANIFOLDS

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Abstract. We present an inductive strategy to show the existence of rational curves on compact Kähler manifolds which are not minimal models but have a pseudoeffective canonical bundle. The tool for this inductive strategy is a weak subadjunction formula for lc centres associated to certain big cohomology classes. This subadjunction formula is based, as in the projective case, on positivity arguments for relative adjoint classes.

1. Introduction

1.A. **Main results.** Rational curves have played an important role in the classification theory of projective manifolds ever since Mori showed that they appear as a geometric obstruction to the nefness of the canonical bundle.

1.1. Theorem. [Mor79, Mor82] *Let X be a complex projective manifold such that the canonical bundle K_X is not nef. Then there exists a rational curve $C \subset X$ such that $K_X \cdot C < 0$.*

Although it is likely that this statement also holds for (non-algebraic) compact Kähler manifolds, Mori's proof which uses a reduction to positive characteristic in an essential way and thus does not adapt to this more general setting. The recent progress on the MMP for Kähler threefolds [HP13] makes crucial use of results on deformation theory of curves on threefolds which are not available in higher dimension. The aim of this paper is to establish an inductive approach to the existence of rational curves. Our starting point is the following

1.2. Conjecture. *Let X be a compact Kähler manifold. Then the canonical class K_X is pseudoeffective if and only if X is not uniruled (i.e. not covered by rational curves).*

This conjecture is shown for projective manifolds in [BDPP13] and it is also known in dimension three by a theorem of Brunella [Bru06] using his theory of rank one foliations. Our main result is as follows:

1.3. Theorem. *Let X be a compact Kähler manifold of dimension n. Suppose that Conjecture 1.2 holds for all manifolds of dimension at most $n-1$. If K_X is pseudoeffective but not nef, there exists a K_X-negative rational curve $f : \mathbb{P}^1 \to X$.*

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Our statement is actually a bit more precise: the K_X-negative rational curve has zero intersection with a cohomology class that is nef and big, so the class of the curve lies in an extremal face of the (generalised) Mori cone. So far we do not know if there exists a morphism contracting this extremal face.

In low dimension we can combine our theorem with Brunella's result:

1.4. Corollary. *Let X be a compact Kähler manifold of dimension at most four. If K_X is pseudoeffective but not nef, there exists a rational curve $f : \mathbb{P}^1 \to X$ such that $K_X \cdot f(\mathbb{P}^1) < 0$.*

1.B. **The strategy.** The idea of the proof is quite natural and inspired by well-known results of the minimal model program: let X be a compact Kähler manifold such that K_X is pseudoeffective but not nef. We choose a Kähler class ω such that $\alpha := K_X + \omega$ is nef and big but not Kähler. If we suppose that X is projective and ω is an $\mathbb{R}$-divisor class we know by the base point free theorem [HM05, Thm.7.1] that there exists a morphism

$$\mu : X \to X'$$

such that $\alpha = \mu^* \omega'$ with ω' an ample $\mathbb{R}$-divisor class on X'. Since α is big the morphism φ is birational, and we denote by Z an irreducible component of its exceptional locus. A general fibre F of $Z \to \mu(Z)$ has positive dimension and is covered by rational curves [MM86, Thm.5], in particular Z is uniruled.

If X is Kähler we are far from knowing the existence of a contraction, however we can still consider the null-locus

$$\mathrm{Null}(\alpha) = \bigcup_{\int_Z \alpha|_Z^{\dim Z} = 0} Z.$$

It is easy to see that if a contraction theorem holds also in the Kähler setting, then the null-locus is exactly the exceptional locus of the bimeromorphic contraction μ. Since the contraction morphism μ is a projective map we could thus apply [MM86, Thm.5] to see that the null locus is uniruled. In this paper we prove directly that at least one of the irreducible components $Z \subset \mathrm{Null}(\alpha)$ is covered by α-trivial rational curves if the Conjecture 1.2 holds.

Let $\pi : Z' \to Z$ be a desingularisation, and let k be the numerical dimension of $\pi^* \alpha|_Z$ (cf. Definition 2.5). Assume for the moment that the contraction μ exists: since Z is in the null-locus we have $k < \dim Z$ and the cohomology class $\pi^* \alpha|_Z^k$ is represented by some multiple of F where F is an irreducible component of a general fibre of $Z \to \mu(Z)$. Since F is an irreducible component of a μ-fibre the conormal sheaf is "semipositive", so we expect that

$$K_{F'} \cdot (\pi|_{F'})^* \omega|_F^{\dim Z - k - 1} \leq (\pi|_{F'})^* K_X|_F \cdot (\pi|_{F'})^* \omega|_F^{\dim Z - k - 1} \tag{1}$$

where $\pi|_{F'} : F' \to F$ is the desingularisation induced by π. Since $\alpha|_F$ is trivial and $\alpha = K_X + \omega$ we see that the right hand side is negative, in particular $K_{F'}$ is not pseudoeffective. Since F is general we obtain that $K_{Z'}$ is not pseudoeffective and we conclude by applying Conjecture 1.2.

Without assuming the existence of μ we will establish a numerical version of (1), i.e. we will prove that

$$K_{Z'} \cdot \pi^* \alpha|_Z^k \cdot \pi^* \omega|_Z^{\dim Z - k - 1} \leq \pi^* K_X|_Z \cdot \pi^* \alpha|_Z^k \cdot \pi^* \omega|_Z^{\dim Z - k - 1}. \tag{2}$$

Note that the right hand side is negative, so Conjecture 1.2 yields the theorem. The inequality (2) follows from a more general weak subadjunction formula for maximal lc centres (cf. Definition 4.4) of the pair $(X, c\alpha)$ (for some real number $c > 0$) which we will explain in the next section. The idea of seeing the irreducible components of the null locus as an lc centre for a suitably chosen pair is already present in Takayama's proof of uniruledness of stable base loci ([Tak08, BBP13]), in our case a recent result of Collins and Tosatti [CT13, Thm.1.1] and the work of Boucksom [Bou04] yield this property without too much effort.

While (2) and Conjecture 1.2 imply immediately that Z is uniruled it is not obvious if we can choose the rational curves to be α-trivial. If Z' was projective and $\pi^*\alpha|_Z$ an $\mathbb{R}$-divisor class we could argue as in [HP13, Prop.7.12] using Araujo's description of the mobile cone [Ara10, Thm.1.3]. In the Kähler case we need a new argument: let $Z' \to Y$ be the MRC-fibration (cf. Remark 6.10) and let F be a general fibre. Arguing by contradiction we suppose that F is not covered by α-trivial rational curves. Using a positivity theorem for relative adjoint classes (Theorem 5.4) we know that $K_{Z'/Y} + \pi^*\alpha|_Z$ is pseudoeffective if $K_F + (\pi^*\alpha|_Z)|_F$ is pseudoeffective. Since $(\pi^*\alpha|_Z)|_F$ is nef the last property is likely to hold if we replace $(\pi^*\alpha|_Z)|_F$ by $\lambda(\pi^*\alpha|_Z)|_F$ for some $\lambda \gg 0$. Somewhat surprisingly this is not quite trivial and leads to an interesting technical problem (Problem 6.4) related to the Nakai-Moishezon criterion for $\mathbb{R}$-divisors by Campana and Peternell [CP90]. Using the minimal model program for the projective manifold F and Kawamata's bound on the length of extremal rays [Kaw91, Thm.1] we overcome this problem in Proposition 6.9. Applying Conjecture 1.2 one more time to the base Y we finally obtain that $K_{Z'} + \lambda\pi^*\alpha|_Z$ is pseudoeffective for some $\lambda \gg 0$. In view of (2) this yields the main theorem.

1.C. **Weak subadjunction.** Let X be a complex projective manifold, and let Δ be an effective $\mathbb{Q}$-Cartier divisor on X such that the pair (X, Δ) is log-canonical. Then there is a finite number of log-canonical centres associated to (X, Δ) and if we choose $Z \subset X$ an lc centre that is minimal with respect to the inclusion, the Kawamata subadjunction formula holds [Kaw98] [FG12, Thm1.2]: the centre Z is a normal variety and there exists a boundary divisor Δ_Z such that (Z, Δ_Z) is klt and

$$K_Z + \Delta_Z \sim_{\mathbb{Q}} (K_X + \Delta)|_Z.$$

If the centre Z is not minimal the geometry is more complicated, however we can still find an *effective* $\mathbb{Q}$-divisor $\Delta_{\tilde{Z}}$ on the normalisation $\nu : \tilde{Z} \to Z$ such that[1]

$$K_{\tilde{Z}} + \Delta_{\tilde{Z}} \sim_{\mathbb{Q}} \nu^*(K_X + \Delta)|_Z.$$

We prove a weak analogue of the subadjunction formula for cohomology classes:

1.5. Theorem. *Let X be a compact Kähler manifold, and let α be a cohomology class on X that is a modified Kähler class (cf. Definition 4.1). Suppose that $Z \subset X$ is a maximal lc centre of the pair (X, α), and let $\nu : \tilde{Z} \to Z$ be the normalisation. Then we have*

$$K_{\tilde{Z}} \cdot \omega_1 \cdot \ldots \cdot \omega_{\dim Z - 1} \leq \nu^*(K_X + \alpha)|_Z \cdot \omega_1 \cdot \ldots \cdot \omega_{\dim Z - 1},$$

where $\omega_1, \ldots, \omega_{\dim Z-1}$ are arbitrary nef classes on $\tilde{Z}$.

[1] This statement is well-known to experts, cf. [BHN14, Lemma 3.1] for a proof.

Our proof follows the strategy of Kawamata in [Kaw98]: given a log-resolution $\mu : \tilde{X} \to X$ and an lc place E_1 dominating Z we want to use a canonical bundle formula for the fibre space $\mu|_{E_1} : E_1 \to \tilde{Z}$ to relate $\mu^*(K_X + \alpha)|_{E_1}$ and $K_{\tilde{Z}}$. As in [Kaw98] the main ingredient for a canonical bundle formula is the positivity theorem for relative adjoint classes Theorem 3.3 which, together with Theorem 5.4, is the main technical contribution of this paper. The main tool of the proofs of Theorem 3.3 and Theorem 5.4 is the positivity of the fibrewise Bergman kernel which is established in [BP08, BP10]. Since we work with lc centres that are not necessarily minimal the positivity result Theorem 3.3 has to be stated for pairs which might not be (sub-)klt. This makes the setup of the proof quite heavy, but similar to earlier arguments (cf. [BP10, Pău12b] and [FM00, Tak06] in the projective case).

The following elementary example illustrates Theorem 1.5 and shows how it leads to Theorem 1.3:

1.6. Example. Let X' be a smooth projective threefold, and let $C \subset X'$ be a smooth curve such that the normal bundle $N_{C/X'}$ is ample. Let $\mu : X \to X'$ be the blow-up of X' along C and let Z be the exceptional divisor. Let $D \subset X'$ be a smooth ample divisor containing the curve C, and let D' be the strict transform.

By the adjunction formula we have $K_Z = (K_X + Z)|_Z$, in particular it is not true that $K_Z \cdot \omega_1 \leq K_X|_Z \cdot \omega_1$ for every nef class ω_1 on Z. Indeed this would imply that $-Z|_Z$ is pseudoeffective, hence $N^*_{C/X'}$ is pseudoeffective in contradiction to the construction. However if we set $\alpha := \mu^* c_1(D)$, then α is nef and represented by $\mu^* D = D' + Z$. Then the pair $(X, D' + Z)$ is log-canonical and Z is a maximal lc centre. Moreover we have

$$K_Z \cdot \omega_1 = (K_X + Z)|_Z \cdot \omega_1 \leq (K_X + D' + Z)|_Z \cdot \omega_1 = (K_X + \alpha)|_Z \cdot \omega_1$$

since $D'|_Z$ is an effective divisor.

Now we set $\omega_1 = \alpha|_Z$, then $\alpha|_Z \cdot \omega_1 = \alpha|_Z^2 = 0$ since it is a pull-back from C. Since K_X is anti-ample on the μ-fibres we have

$$K_Z \cdot \alpha|_Z = K_X|_Z \cdot \alpha|_Z < 0.$$

Thus K_Z is not pseudoeffective.

1.D. **Relative adjoint classes.** We now explain briefly the idea of the proof of Theorem 3.3 and Theorem 5.4. In view of the main results in [BP08] and [Pău12a], it is natural to ask the following question :

1.7. Question. *Let X and Y be two compact Kähler manifolds of dimension m and n respectively, and let $f : X \to Y$ be a surjective map with connected fibres. Let F be the general fiber of f. Let α_X be a Kähler class on X and let D be a klt $\mathbb{Q}$-divisor on X such that $c_1(K_F) + [(\alpha_X + D)|_F]$ is a pseudoeffective class. Is $c_1(K_{X/Y}) + [\alpha_X + D]$ pseudoeffective ?*

In the case $D = 0$ and $c_1(K_F) + [\alpha_X|_F]$ is a Kähler class on F, [Pău12a] confirm the above question by studying the variation of Kähler-Einstein metrics (based on [Sch12]). In our article, we confirm Question 1.7 in two special cases: Theorem 3.3 and Theorem 5.4 by using the positivity of the fibrewise Bergman kernel which is established in [BP08, BP10]. Let us compare our results to Păun's result [Pău12a, Thm.1.1] on relative adjoint classes: while we make much weaker assumptions on the geometry of pairs or the positivity of the involved cohomology classes we are always in a situation where locally over the base we only have to deal with $\mathbb{R}$-divisor classes. Thus the transcendental character of the argument is only apparent on the base, not along the general fibres.

More precisely, in Theorem 3.3, we add an additional condition that $c_1(K_{X/Y} + [\alpha_X + D])$ is pull-back of a $(1,1)$-class on Y (but we assume that D is sub-boundary). Then we can take a Stein cover (U_i) of Y such that $(K_{X/Y} + [\alpha_X + D])|_{f^{-1}(U_i)}$ is trivial on $f^{-1}(U_i)$. Therefore $[\alpha_X + D]|_{f^{-1}(U_i)}$ is a $\mathbb{R}$-line bundle on $f^{-1}(U_i)$. We assume for simplicity that D is klt (the sub-boundary case is more complicated). We can thus apply [BP10] to every pair $(f^{-1}(U_i), K_{X/Y} + [\alpha_X + D])$. Since the fibrewise Bergman kernel metrics are defined fiber by fiber, by using $\partial\overline{\partial}$-lemma, we can glue the metrics together and Theorem 3.3 is thus proved.

In Theorem 5.4, we add the condition that F is simply connected and $H^0(F, \Omega_F^2) = 0$ [2]. Then we can find a Zariski open set Y_0 of Y such that $R^i f_*(\mathcal{O}_X) = 0$ on Y_0 for every $i = 1, 2$. By using the same argument as in Theorem 3.3, we can construct a quasi-psh function φ on $f^{-1}(Y_0)$ such that $\frac{\sqrt{-1}}{2\pi}\Theta(K_{X/Y}) + \alpha_X + dd^c\varphi \geq 0$ on $f^{-1}(Y_0)$. Now the main problem is to extend φ to be a quasi-psh function on X. Since $c_1(K_F + \alpha_X|_F)$ is not necessary a Kähler class on F, we cannot use directly the method in [Pău12a, 3.3] . Here we use the idea in [LAE02]. In fact, thanks to [LAE02, Part II, Thm 1.3], we can find an increasing sequence $(k_m)_{m\in\mathbb{N}}$ and hermitian line bundles $(F_m, h_m)_{m\in\mathbb{N}}$ (not necessarily holomorphic) on X such that

$$\|\frac{\sqrt{-1}}{2\pi}\Theta_{h_m}(F_m) - k_m(\frac{\sqrt{-1}}{2\pi}\Theta(K_{X/Y}) + \alpha_X)\|_{C^\infty(X)} \to 0. \tag{3}$$

Let X_y be the fiber over $y \in Y_0$. As we assume that $H^0(X_y, \Omega^2_{X_y}) = 0$, $F_m|_{X_y}$ is a holomorphic line bundle on X_y. Therefore we can define the Bergman kernel metric associated to $(F_m|_{X_y}, h_m)$. Thanks to $\partial\overline{\partial}$-lemma, we can compare $\varphi|_{X_y}$ and the Bergman kernel metric associated to $(F_m|_{X_y}, h_m)$. Note that (3) implies that F_m is more and more holomorphic. Therefore, by using standard Ohsawa-Takegoshi technique [BP10], we can well estimate the Bergman kernel metric associated to $F_m|_{X_y}$ when $y \to Y \setminus Y_0$. Theorem 5.4 is thus proved by combining these two facts.

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2. Notation and terminology

For general definitions we refer to [Har77, KK83, Dem12]. Manifolds and normal complex spaces will always be supposed to be irreducible. A fibration is a proper surjective map with connected fibres $\varphi : X \to Y$ between normal complex spaces.

2.1. Definition. *Let X be a normal complex space, and let $f : X \to Y$ be a proper surjective morphism. A $\mathbb{Q}$-divisor D is f-vertical if $f(\operatorname{Supp} D) \subsetneq Y$. Given a $\mathbb{Q}$-divisor D it admits a unique decomposition*

$$D = D_{f\text{-hor}} + D_{f\text{-vert}}$$

such that $D_{f\text{-vert}}$ is f-vertical and every irreducible component $E \subset \operatorname{Supp} D_{f\text{-hor}}$ surjects onto Y.

[2] If F is rational connected these two conditions are satisfied.

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2.2. Definition. *Let X be a complex manifold, and let $\mathcal{F}$ be a sheaf of rank one on X that is locally free in codimension one. The bidual $\mathcal{F}^{**}$ is reflexive of rank one, so locally free, and we set $c_1(\mathcal{F}) := c_1(\mathcal{F}^{**})$.*

Throughout this paper we will use positivity properties of real cohomology classes of type $(1,1)$, that is elements of the vector space $H^{1,1}(X) \cap H^2(X, \mathbb{R})$. The definitions can be adapted to the case of a normal compact Kähler space X by using Bott-Chern cohomology for $(1,1)$-forms with local potentials [HP13]. In order to simplify the notation we will use the notation

$$N^1(X) := H^{1,1}(X) \cap H^2(X, \mathbb{R}).$$

Note that for the purpose of this paper we will only use cohomology classes that are pull-backs of nef classes on some smooth space, so it is sufficient to give the definitions in the smooth case.

2.3. Definition. [Dem12, Defn 6.16] *Let (X, ω_X) be a compact Kähler manifold, and let $\alpha \in N^1(X)$. We say that α is nef if for every $\epsilon > 0$, there is a smooth $(1,1)$-form α_ϵ in the same class of α such that $\alpha_\epsilon \geq -\epsilon\omega_X$.*
We say that α is pseudoeffective if there exists a $(1,1)$-current $T \geq 0$ in the same class of α. We say that α is big if there exists a $\epsilon > 0$ such that $\alpha - \epsilon\omega_X$ is pseudoeffective.

2.4. Definition. *Let X be a compact Kähler manifold, and let $\alpha \in N^1(X)$ be a nef and big cohomology class on X. The null-locus of α is defined as*

$$\mathit{Null}(\alpha) = \bigcup_{\int_Z \alpha|_Z^{\dim Z} = 0} Z.$$

Remark. A priori the null-locus is a countable union of proper subvarieties of X. However by [CT13, Thm.1.1] the null-locus coincides with the non-Kähler locus $E_{nK}(\alpha)$, in particular it is an analytic subvariety of X.

2.5. Definition. [Dem12, Defn 6.20] *Let X be a compact Kähler manifold, and let $\alpha \in N^1(X)$ be a nef class. We define the numerical dimension of α by*

$$\mathrm{nd}(\alpha) := \max\{k \in \mathbb{N} \mid \alpha^k \neq 0 \textit{ in } H^{2k}(X, \mathbb{R})\}.$$

2.6. Remark. A nef class α is big if and only if $\int_X \alpha^{\dim X} > 0$ [DP04, Thm.0.5] which is of course equivalent to $\mathrm{nd}(\alpha) = \dim X$.
By [Dem12, Prop 6.21] the cohomology class $\alpha^{\mathrm{nd}(\alpha)}$ can be represented by a non-zero closed positive $(\mathrm{nd}(\alpha), \mathrm{nd}(\alpha))$-current T. Therefore $\int_X \alpha^{\mathrm{nd}(\alpha)} \wedge \omega_X^{\dim X - \mathrm{nd}(\alpha)} > 0$ for any Kähler class ω_X.

2.7. Definition. *Let X be a normal compact complex space of dimension n, and let $\omega_1, \ldots, \omega_{n-1} \in N^1(X)$ be cohomology classes. Let $\mathcal{F}$ be a reflexive rank one sheaf on X, and let $\pi : X' \to X$ be a desingularisation. We define the intersection number $c_1(\mathcal{F}) \cdot \omega_1 \cdot \ldots \cdot \omega_{n-1}$ by*

$$c_1((\mu^*\mathcal{F})^{**}) \cdot \mu^*\omega_1 \cdot \ldots \cdot \mu^*\omega_{n-1}.$$

Remark. The definition above does not depend on the choice of the resolution π: the sheaf $\mathcal{F}$ is reflexive of rank one, so locally free on the smooth locus of X. Thus $\mu^*\mathcal{F}$ is locally free in the complement of the μ-exceptional locus. Thus $\pi_1 : X_1' \to X$ and $\pi_2 : X_2' \to X$ are two resolutions and Γ is a manifold dominating X_1' and X_2'

via bimeromorphic morphisms q_1 and q_2, then $q_1^*\pi_1^*\mathcal{F}$ and $q_2^*\pi_2^*\mathcal{F}$ coincide in the complement of the $\pi_1 \circ q_1 = \pi_2 \circ q_2$-exceptional locus. Thus their biduals coincide in the complement of this locus. By the projection formula their intersection with classes coming from X are the same.

3. Positivity of relative adjoint classes, part 1

Before the proof of the main theorem in this section, we first recall the construction of fibrewise Bergman kernel metric and its important property, which are established in the works [BP08] and [BP10]. The original version ([BP10, Thm 0.1]) concerns only the projective fibration. However, thanks to the optimal extension theorem [GZ15], we know that it is also true for the Kähler case :

3.1. Theorem. [BP10, Thm 0.1], [GZ15, 3.5], [Cao14, Thm 1.2] *Let $p : X \to Y$ be a proper fibration between Kähler manifolds of dimension m and n respectively, and let L be a line bundle endowed with a metric h_L such that:*

1) The curvature current of the bundle (L, h_L) is semipositive in the sense of current, i.e., $\sqrt{-1}\Theta_{h_L}(L) \geq 0$;

2) there exists a general point $z \in Y$ and a section $u \in H^0(X_z, mK_{X_z} + L)$ such that

$$\int_{X_z} |u|_{h_L}^{\frac{2}{m}} < +\infty.$$

Then the line bundle $mK_{X/Y} + L$ admits a metric with positive curvature current. Moreover, this metric is equal to the fibrewise m-Bergman kernel metric on the general fibre of p.

3.2. Remark. The fibrewise m-Bergman kernel metric is defined as follows : Let $x \in X$ be a point on a smooth fibre of p. We first define a hermitian metric h on $-(mK_{X/Y} + L)_x$ by

$$\|\xi\|_h^2 := \sup \frac{|\tau(x) \cdot \xi|^2}{(\int_{X_{p(x)}} |\tau|_{h_L}^{\frac{2}{m}})^m},$$

where the 'sup' is taken over all sections $\tau \in H^0(X_{p(x)}, mK_{X/Y} + L)$. The fibrewise m-Bergman kernel metric on $mK_{X/Y} + L$ is defined to be the dual of h.

It will be useful to give a more explicit expression of the Bergman kernel type metric. Let ω_X and ω_Y be Kähler metrics on X and Y respectively. Then ω_X and ω_Y induce a natural metric $h_{X/Y}$ on $K_{X/Y}$. Let Y_0 be a Zariski open set of Y such that p is smooth over Y_0. Set $h_0 := h_{X/Y}^m \cdot h_L$ be the induced metric on $mK_{X/Y} + L$. Let φ be a function $p^{-1}(Y_0)$ defined by

$$\varphi(x) = \sup_{\tau \in A} \frac{1}{m} \ln |\tau|_{h_0}(x),$$

where

$$A := \{f \mid f \in H^0(X_{p(x)}, mK_{X/Y} + L) \text{ and } \int_{X_{p(x)}} |f|_{h_0}^{\frac{2}{m}} (\omega_X^m / p^*\omega_Y^n) = 1\}.$$

We can easily check that the metric $h_0 \cdot e^{-2m\varphi}$ on $mK_{X/Y} + L$ coincides with the fibrewise m-Bergman kernel metric defined above. In particular, $h_0 \cdot e^{-2m\varphi}$ is independent of the choice of the metrics ω_X and ω_Y. Sometimes we call φ the fibrewise m-Bergman kernel metric.

Proof. We recall briefly the idea of the proof of Theorem 3.1. Note first that the fibrewise m-Bergman kernel metric $h_B := (h_{X/Y})^m \cdot h_L \cdot e^{-2m\varphi}$ is well defined only on an open set $p^{-1}(Y_0)$ of X, where Y_0 is a Zariski open set of Y. If p is projective, by using [Ber09, Thm 1.2] and Demailly's regularization techniques, [BP10, Thm 0.1] proved first that

$$\sqrt{-1}\Theta_{h_B}(mK_{X/Y} + L) \geq 0 \qquad \text{on } p^{-1}(Y_0). \tag{4}$$

In the case p is not necessary projective, we can prove (4) by using optimal extension theorem [GZ15, 3.5] (cf. also [Cao14, Thm 1.2] for the Kähler case). We remark that both two approaches are local estimate with respect to the base manifold Y. Finally, by using standard Ohsawa-Takegoshi extension theorem, [BP10, Thm 0.1] proved that the quasi-psh function φ defined on $p^{-1}(Y_0)$ can be extended to be a quasi-psh function on the total space X and satisfies also

$$\sqrt{-1}\Theta_{h_B}(mK_{X/Y} + L) \geq 0 \qquad \text{on } X.$$

The theorem is thus proved. □

Here is the main theorem in this section.

3.3. Theorem. *Let X and Y be two compact Kähler manifolds of dimension m and n respectively, and let $f : X \to Y$ be a surjective map with connected fibres. Let α_X be a Kähler class on X. Let*[4] *$D = \sum_{j=2}^{k} -d_j D_j$ be a $\mathbb{Q}$-divisor on X such that the support has simple normal crossings. Suppose that the following properties hold:*

(a) *If $d_j \leq -1$ then $f(D_j)$ has codimension at least 2.*
(b) *The direct image sheaf $f_*\mathcal{O}_X(\lceil -D \rceil)$ has rank one. Moreover, if $D = D^h + D^v$ is the decomposition in a f-horizontal part D^h (resp. f-vertical part D^v) then we have $(f_*\mathcal{O}_X(\lceil -D^v \rceil))^{**} \simeq \mathcal{O}_Y$.*
(c) *$K_{X/Y} + \alpha_X + D = f^*L$ for some class $L \in N^1(Y)$.*

Let $\omega_1, \omega_2, \cdots, \omega_{\dim Y - 1}$ be nef classes on Y. Then we have

$$L \cdot \omega_1 \cdots \omega_{\dim Y - 1} \geq 0. \tag{5}$$

Proof. Step 1: Preparation.

We start by interpreting the conditions (a) and (b) in a more analytic language. We can write the divisor D as

$$D = B - F^v - F^h,$$

where B, F^v, F^h are effective $\mathbb{Q}$-divisors and F^v (resp. F^h) is f-vertical (resp. f-horizontal). We also decompose F^v as

$$F^v = F_1^v + F_2^v$$

such that $\operatorname{codim}_Y f(F_2^v) \geq 2$ and $\operatorname{codim}_Y f(E) = 1$ for every irreducible component $E \subset F_1^v$.

[4] The somewhat awkward notation will be become clear in the proof of Theorem 1.5.

Let X_y be a general f-fibre. Since $d_j > -1$ for every D_j mapping onto Y (cf. condition (a)), the divisors $\lceil -D \rceil$ and $\lceil F^h \rceil$ coincide over a non-empty Zariski open subset of Y. Thus the condition rank $f_* \mathcal{O}_X(\lceil -D \rceil) = 1$ implies that

$$h^0(X_y, \lceil F^h \rceil|_{X_y}) = 1.$$

Therefore, for any meromorphic function ζ on X_y, we have

$$\text{div}(\zeta) \geq -\lceil F^h \rceil|_{X_y} \qquad \Rightarrow \qquad \zeta \text{ is constant.} \tag{6}$$

Since $d_j > -1$ for every D_j mapping onto a divisor in Y (cf. condition (a)), the divisors $\lceil -D^v \rceil$ and $\lceil F^v \rceil$ coincide over a Zariski open subset $Y_1 \subset Y$ such that $\text{codim}_Y(Y \setminus Y_1) \geq 2$. In particular the condition $(f_* \mathcal{O}_X(\lceil -D^v \rceil))^{**} \simeq \mathcal{O}_Y$ implies that $(f_* \mathcal{O}_X(\lceil -D^v \rceil))|_{Y_1} = \mathcal{O}_{Y_1}$. So for every meromorphic function ζ on any small Stein open subset of $U \subset Y_1$, we have

$$\text{div}(\zeta \circ f) \geq -\lceil F^v \rceil|_{f^{-1}(U)} \qquad \Rightarrow \qquad \zeta \text{ is holomorphic.} \tag{7}$$

Step 2: Stein cover.

Fix a Kähler metric ω_X (respectively ω_Y) on X (respectively on Y). Select a Stein cover $(U_i)_{i \in I}$ of Y such that $H^{1,1}(U_i, \mathbb{R}) = 0$ for every i. Let h be the smooth metric on $K_{X/Y}$ induced by ω_X and ω_Y. Set $\beta := \frac{\sqrt{-1}}{2\pi} \Theta_h(K_{X/Y})$. Thanks to (c), we have $(\beta + \alpha_X + D)|_{f^{-1}(U_i)} \in f^{-1}(H^{1,1}(U_i, \mathbb{R})) = 0$. By the Lefschetz $(1,1)$-theorem there exists a line bundle L_i on $f^{-1}(U_i)$ with a singular hermitian metric h_i such that $K_{X/Y} + L_i$ is a trivial line bundle on $f^{-1}(U_i)$ and

$$\frac{\sqrt{-1}}{2\pi} \Theta_h(K_{X/Y}) + \frac{\sqrt{-1}}{2\pi} \Theta_{h_i}(L_i) = \beta + \alpha_X + D \qquad \text{on } f^{-1}(U_i). \tag{8}$$

Step 3: Local construction of metric.

We construct in this step a 'canonical' function φ_i on $f^{-1}(U_i)$ such that

$$\alpha_X + \beta + D + dd^c \varphi_i \geq -\lceil F_2^v \rceil \qquad \text{over } f^{-1}(U_i) \qquad \text{for every } i. \tag{9}$$

The function is in fact just the potential of the fibrewise Bergman kernel metric mentioned in Remark 3.2. A more explicit construction is as follows:

Note first that the restriction of $K_{X/Y} + L_i$ on the generic fibre of f is trivial. Combining this with the sub-klt condition (a) and the construction of the metric h_i, we can find a Zariski open subset $U_{i,0}$ of U_i such that for every $y \in U_{i,0}$, f is smooth over y and there exists a $s_y \in H^0(X_y, K_{X/Y} + L_i)$ such that

$$\int_{X_y} |s_y|^2_{h,h_i} (\omega_X^m / f^*(\omega_Y)^n) = 1. \tag{10}$$

Using the fact that

$$h^0(X_y, K_{X/Y} + L_i) = h^0(X_y, \mathcal{O}_{X_y}) = 1 \qquad \text{for every } y \in U_{i,0}, \tag{11}$$

we know that s_y is unique after multiplying by a unit norm complex number. There exists thus a unique function φ_i on $f^{-1}(U_{i,0})$ such that its restriction on X_y equals to $\ln |s_y|_{h,h_i}$. We have the following key property.

Claim: φ_i can be extended to be a function (we still denote it as φ_i) on $f^{-1}(U_i)$, and satisfies (9).

The claim will be proved by using the methods in [BP08, Thm 0.1]. Since L_i is not necessary effective in our case this requires a some more effort. We postpone the proof of the claim later and first finish the proof of the theorem. The properties (6) and (7) will be used in the proof of the claim.

Step 4: Gluing process, final conclusion.
We first prove that

$$\varphi_i = \varphi_j \qquad \text{on } f^{-1}(U_i \cap U_j). \tag{12}$$

Let $y \in U_{i,0} \cap U_{j,0}$. Since both $(K_{X/Y} + L_i)|_{X_y}$ and $(K_{X/Y} + L_j)|_{X_y}$ are trivial line bundles on X_y, we have $L_i|_{X_y} \simeq L_j|_{X_y}$. Under this isomorphism, the curvature condition (8) and $\partial\overline{\partial}$-lemma imply that

$$h_i|_{X_y} = h_j|_{X_y} \cdot e^{-c_y} \qquad \text{for some constant } c_y \text{ on } X_y, \tag{13}$$

where the constant c_y depends on $y \in Y$. By (11), there exist unique elements $s_{y,i} \in H^0(X_y, K_{X/Y} + L_i)$ and $s_{y,j} \in H^0(X_y, K_{X/Y} + L_j)$ (after multiply by a unit norm complex number) such that

$$\int_{X_y} |s_{y,i}|^2_{h_i} (\omega_X^m / f^*(\omega_Y)^n) = 1 \qquad \text{and} \qquad \int_{X_y} |s_{y,j}|^2_{h_j} (\omega_X^m / f^*(\omega_Y)^n) = 1.$$

Thanks to (13), we know that $|s_{y,i}| = e^{\frac{c_y}{2}} |s_{y,j}|$. Therefore

$$\varphi_i|_{X_y} = \ln |s_{y,i}|_{h,h_i} = \ln |s_{y,j}|_{h,h_j} = \varphi_j|_{X_y}. \tag{14}$$

Since (14) is proved for every $y \in U_{i,0} \cap U_{j,0}$, we have $\varphi_i = \varphi_j$ on $f^{-1}(U_{i,0} \cap U_{j,0})$. Combining this with the extension property of quasi-psh functions, (12) is thus proved.

Thanks to (12), $(\varphi_i)_{i \in I}$ defines a global quasi-psh function on X which we denote by φ. By (9), we have

$$(\alpha_X + \beta + D) + dd^c \varphi \geq -\lceil F_2^v \rceil \qquad \text{over } f^{-1}(U_i) \qquad \text{for every } i.$$

Therefore

$$(\alpha_X + \beta + D) + dd^c \varphi \geq -\lceil F_2^v \rceil \qquad \text{over } X.$$

Note that $\operatorname{codim}_Y f_*(F_2^v) \geq 2$, Theorem 3.3 is thus proved. □

The rest part of this section is devoted to the proof of the claim in Theorem 3.3. The main method is the Ohsawa-Takegoshi extension techniques used in [BP10]. Before the proof of the claim, we need the following lemma which interprets the property (7) in terms of a condition on the metric h_i.

3.4. Lemma. *Fix a smooth metric h_0 on L_i over $f^{-1}(U_i)$. Let ψ be the function such that $h_i = h_0 \cdot e^{-2\psi}$. Let Y_1 be the open set defined in Step 1 of the proof of Theorem 3.3 and let $Y_0 \subset Y_1$ be a non-empty Zariski open set satisfying the following conditions :*

(a) *f is smooth over Y_0;*
(b) *$f(D^v) \subset Y \setminus Y_0$;*
(c) *$F^h|_{X_y}$ is snc for every $y \in Y_0$;*
(d) *The property (6) holds for every $y \in Y_0$.*

Then for any open set $\Delta \Subset Y_1 \cap U_i$ (i.e., the closure of Δ is in $Y_1 \cap U_i$), there exists some constant $C(\Delta, Y_1, U_i) > 0$ depending only on Δ, Y_1 and U_i, such that

$$\int_{X_y} e^{-2\psi} \omega_X^m / f^* \omega_Y^n \geq C(\Delta, Y_1, U_i) \qquad \text{for every } y \in \Delta \cap Y_0. \tag{15}$$

3.5. Remark. The meaning of (15) is that, for any sequence $(y_i)_{i\geq 1}$ converging to a point in $Y_1 \setminus Y_0$, the sequence $(\int_{X_{y_i}} e^{-2\psi}\omega_X^m/f^*\omega_Y^n)_{i\geq 1}$ will not tend to 0.

Proof. Note first that, by (8), we have

$$\psi = \ln|B| - \ln|F^v| - \ln|F^h| + C^\infty. \tag{16}$$

Fix an open set Δ_1 such that $\Delta \Subset \Delta_1 \Subset Y_1 \cap U_i$. Let y_0 be a point in $\Delta \cap Y_0$ and let s_{y_0} be a constant such that

$$|s_{y_0}|^2 \int_{X_{y_0}} e^{-2\psi}\omega_X^m/f^*\omega_Y^n = 1. \tag{17}$$

By applying Ohsawa-Takegoshi extension theorem (cf. [Dem12, Thm 12.6]) to $(f^{-1}(\Delta_1) \setminus (F^v \cup F^h), K_X + L_i, h_i)$, we can find a holomorphic function τ on $f^{-1}(\Delta_1) \setminus (F^v \cup F^h)$ (recall that $K_X + L_i$ is a trivial line bundle), such that

$$\tau|_{X_{y_0}} = s_{y_0}$$

and

$$\int_{f^{-1}(\Delta_1)\setminus(F^v\cup F^h)} |\tau|^2_{h,h_0} e^{-2\psi}\omega_X^m \leq C_1|s_{y_0}|^2 \int_{X_{y_0}} e^{-2\psi}\omega_X^m/f^*\omega_Y^n = C_1 \tag{18}$$

where C_1 is a constant independent of $y_0 \in \Delta \cap Y_0$. Then τ can be extended to a meromorphic function $\widetilde{\tau}$ on $f^{-1}(\Delta_1)$ satisfying the same estimate (18). Thanks to (16) and (18), we have

$$\operatorname{div}(\widetilde{\tau}) \geq -\lceil F^h \rceil - \lceil F^v \rceil \qquad \text{on } f^{-1}(\Delta_1). \tag{19}$$

We now prove that $\widetilde{\tau}$ is in fact holomorphic on $f^{-1}(\Delta_1)$. For every point $y \in \Delta_1 \cap Y_0$, we know that $F^v \cap X_y = \emptyset$. Combining this with (19), we can apply (6) to $\widetilde{\tau}|_{X_y}$. As a consequence, $\widetilde{\tau}$ is constant on X_y for every $y \in \Delta_1 \cap Y_0$. Therefore $\widetilde{\tau}$ comes from a meromorphic function on Δ_1. Then $\widetilde{\tau}$ does not have poles along the horizontal direction and (19) implies that

$$\operatorname{div}(\widetilde{\tau}) \geq -\lceil F^v \rceil.$$

Now we can apply (7) to $\widetilde{\tau}$. There exists thus a holomorphic function ζ on Δ_1 such that $\widetilde{\tau} = \zeta \circ f$.

Since ζ is holomorphic on Δ_1 and $\Delta \Subset \Delta_1$, by applying maximal principal to ζ, (18) implies that $\sup_{z\in\Delta} |\zeta|(z) \leq \sqrt{C_1} \cdot C_2$ where C_1 is the constant in (18) and C_2 is a constant depending only on Δ and Δ_1. In particular, $s_{y_0} = \tau|_{X_{y_0}} = \zeta(y_0)$ is controlled by $\sqrt{C_1} \cdot C_2$. Combining this with (17) and the fact that C_1 and C_2 are independent of the choice of $y_0 \in \Delta$, the lemma is proved. □

3.6. Remark. We can also see the estimate (18) by the following more standard argument. Fix a smooth metric g_0 on the line bundle $\lceil F^v \rceil + \lceil F^h \rceil$. Let u be a canonical section of $\lceil F^v \rceil + \lceil F^h \rceil$. Set $\psi_1 := \ln|u|_{g_0}$ and $g_1 := g_0 \cdot e^{-2\psi_1}$. Then g_1 is a singular metric on $\lceil F^v \rceil + \lceil F^h \rceil$ and we have

$$|u|^2_{g_1}(x) = |u|^2_{g_0}(x) \cdot e^{-2\psi_1(x)} = 1 \qquad \text{for every } x \in f^{-1}(U_i) \tag{20}$$

and

$$\frac{\sqrt{-1}}{2\pi}\Theta_{h_i,g_1}(L_i + \lceil F^v \rceil + \lceil F^h \rceil) = \alpha_X + D + \lceil F^v \rceil + \lceil F^h \rceil \geq 0 \qquad \text{on } f^{-1}(U_i). \tag{21}$$

We can thus apply the standard Ohsawa-Takegoshi extension to the setting

$$(f^{-1}(\Delta_1), K_X + L_i + \lceil F^v \rceil + \lceil F^h \rceil, h_i \cdot g_1).$$

The section $s_{y_0} \cdot u$ on X_{y_0} can be thus extended to a section $S \in H^0(f^{-1}(\Delta_1), K_X + L_i + \lceil F^v \rceil + \lceil F^h \rceil)$ and satisfies

$$\int_{f^{-1}(\Delta_1)} |S|^2_{h,h_i,g_1} \omega_X^m \leq C \int_{X_{y_0}} |s_{y_0} \cdot u|^2_{h,h_i,g_1} \omega_X^m / f^* \omega_Y^n$$

for some uniform constant C. Set $\tau := \frac{S}{u}$. Recall that $K_X + L_i$ is a trivial line bundle on $f^{-1}(U_i)$, thus τ is a meromorphic function on $f^{-1}(\Delta)$. Thanks to (20), we have

$$\int_{f^{-1}(\Delta_1)} |S|^2_{h,h_i,g_1} \omega_X^m = \int_{f^{-1}(\Delta_1)} |\tau|^2_{h,h_i} \cdot |u|^2_{g_1} \omega_X^m = \int_{f^{-1}(\Delta_1)} |\tau|^2_{h,h_i} \omega_X^m.$$

By the same reason, we have

$$\int_{X_{y_0}} |s_{y_0} \cdot u|^2_{h,h_i,g_1} \omega_X^m / f^* \omega_Y^n = \int_{X_{y_0}} |s_{y_0}|^2_{h,h_i} \omega_X^m / f^* \omega_Y^n.$$

Combining the above three equations, we obtain

$$\int_{f^{-1}(\Delta_1)} |\tau|^2_{h,h_i} \omega_X^m \leq C \int_{X_{y_0}} |s_{y_0}|^2_{h,h_i} \omega_X^m / f^* \omega_Y^n \tag{22}$$

Note that h, h_0 are fixed smooth metrics, so (22) implies (18).

Now we prove the claim in the proof of Theorem 3.3.

Proof of the claim. Fix a smooth metric h_s on the line bundle $\lceil F^v \rceil + \lceil F^h \rceil$. Let s_F be a canonical section of $\lceil F^v \rceil + \lceil F^h \rceil$. Let ψ be the function defined in Lemma 3.4. Set $\psi_1 := \ln |s_F|_{h_s}$ and $h_F := h_s \cdot e^{-2\psi_1}$ be the singular metric on $\lceil F^v \rceil + \lceil F^h \rceil$. We have

$$\frac{\sqrt{-1}}{2\pi} \Theta_{h_F}(\lceil F^v \rceil + \lceil F^h \rceil) = \lceil F^v \rceil + \lceil F^h \rceil$$

Then

$$|s_F|^2_{h_F}(x) = |s_F|^2_{h_s}(x) \cdot e^{-2\psi_1(x)} = 1 \qquad \text{for every } x \in f^{-1}(U_i) \tag{23}$$

and

$$\frac{\sqrt{-1}}{2\pi} \Theta_{h_i,h_F}(L_i + \lceil F^v \rceil + \lceil F^h \rceil) = \alpha_X + D + \lceil F^v \rceil + \lceil F^h \rceil \geq 0 \qquad \text{on } f^{-1}(U_i). \tag{24}$$

Let $U_{i,0}$ be the open set defined in Step 3 of the proof of Theorem 3.3. Let $y \in U_{i,0}$ and let $s_y \in H^0(X_y, mK_{X/Y} + L_i)$ such that

$$\int_{X_y} |s_y|^2_{h,h_i} \omega_X^m / f^* \omega_Y^n = 1. \tag{25}$$

Thanks to (23), we have

$$\int_{X_y} |s_y \cdot s_F|^2_{h,h_i,h_F} \omega_X^m / f^* \omega_Y^n = \int_{X_y} |s_y|^2_{h,h_i} \omega_X^m / f^* \omega_Y^n = 1.$$

Let s be the function on $f^{-1}(U_{i,0})$ such that $s|_{X_y} = s_y$ for every $y \in U_{i,0}$. Thanks to (6), for a generic $y \in U_{i,0}$, we have $h^0(X_y, K_{X/Y} + L_i + \lceil F^v \rceil + \lceil F^h \rceil) = 1$.

Therefore $\ln |s\cdot s_F|_{h,h_i,h_F}$ is the Bergman kernel metric (cf. Remark (3.2)) defined over $f^{-1}(U_{i,0})$ with respect to the setting

$$(K_{X/Y}+L_i+\lceil F^v\rceil+\lceil F^h\rceil, h_i\cdot h_F).$$

Note that $h_i\cdot h_F=h_0\cdot h_s\cdot e^{-2\psi-2\psi_1}$. By (24) and Theorem 3.1, we know that $\ln |s\cdot s_F|_{h,h_i,h_F}+\psi_1+\psi$ can be extended as a quasi-psh function on $f^{-1}(U_i)$ and satisfying

$$(\alpha_X+\beta+D+\lceil F^v\rceil+\lceil F^h\rceil)+dd^c\ln |s\cdot s_F|_{h,h_i,h_F}\geq 0 \qquad \text{on } f^{-1}(U_i).$$

Note that by (23), we have

$$\varphi_i=\ln |s|_{h,h_i}=\ln |s\cdot s_F|_{h,h_i,h_F} \qquad \text{on } f^{-1}(U_i\cap Y_0).$$

Therefore $\varphi_i+\psi_1+\psi$ can be extended to be a quasi-psh function on $f^{-1}(U_i)$ and satisfies

$$(\alpha_X+\beta+D+\lceil F^v\rceil+\lceil F^h\rceil)+dd^c\varphi_i\geq 0 \qquad \text{over } f^{-1}(U_i). \tag{26}$$

We next prove that s is uniformly upper bounded near the generic point of $\mathrm{div}(F^h)$ and near the generic point of $\mathrm{div}(F_1^v)$. Let y be a generic point in $U_{i,0}$. By the construction of s_y and (6), s_y is a constant on X_y. Therefore s is uniformly bounded near the generic point of $\mathrm{div}(F^h)$. Moreover, for any $\Delta\Subset Y_1\cap U_i$, thanks to Lemma 3.4, there exists a constant $c>0$, such that

$$\int_{X_y} e^{-2\psi}(\omega_X^m/f^*(\omega_Y)^n)\geq c \qquad \text{for every } y\in\Delta\cap Y_0.$$

Combining this with (25), we see that s_y is uniformly upper bounded on $f^{-1}(\Delta\cap Y_0)$. Since $\mathrm{codim}_Y(Y\setminus Y_1)\geq 2$ and $f_*(F_1^v)$ is of codimension 1 by assumption, the function s is uniformly upper bounded near the generic point of $\mathrm{div}(F_1^v)$.

Now we prove the claim. By the definition of ψ and (16), we have

$$h_i=h_0\cdot e^{-2\psi} \qquad \text{and} \qquad \psi=\ln|B|-\ln|F^v|-\ln|F^h|+C^\infty.$$

Therefore

$$\varphi_i=\ln|s|_{h,h_0}-\psi=\ln|s|_{h,h_0}+\ln|F^v|+\ln|F^h|-\ln|B|+C^\infty.$$

Since s is proved to be uniformly upper bounded near the generic point of $\mathrm{div}(F^h)$ and near the generic point of $\mathrm{div}(F_1^v)$, the Lelong numbers of $dd^c\varphi_i$ at the generic points of $\mathrm{div}(F^h)+\mathrm{div}(F_1^v)$ is not smaller than the Lelong numbers of the current $|F^h+F_1^v|$ at the generic points of $\mathrm{div}(F^h)+\mathrm{div}(F_1^v)$. Combining with the construction of ψ and ψ_1, we know that the Lelong numbers of the current $dd^c(\varphi_i+\psi+\psi_1)$ at the generic points of $\mathrm{div}(F^h)+\mathrm{div}(F_1^v)$ is not smaller than the Lelong numbers of the current $|\lceil F^h\rceil+\lceil F_1^v\rceil|$ at the generic points of $\mathrm{div}(F^h)+\mathrm{div}(F_1^v)$. By definition

$$(\alpha_X+\beta+D+\lceil F^v\rceil+\lceil F^h\rceil)+dd^c\varphi_i=\beta+\frac{\sqrt{-1}}{2\pi}\Theta_{h_0,h_s}(L_i+\lceil F^h\rceil+\lceil F^v\rceil)+dd^c(\varphi_i+\psi+\psi_1)\geq 0$$

Since h_0 and h_s are smooth, the above estimation of the Lelong numbers of the current $dd^c(\varphi_i+\psi+\psi_1)$ implies that

$$\beta+\frac{\sqrt{-1}}{2\pi}\Theta_{h_0,h_s}(L_i+\lceil F^h\rceil+\lceil F^v\rceil)+dd^c(\varphi_i+\psi+\psi_1)\geq\lceil F^h\rceil+\lceil F_1^v\rceil.$$

Therefore

$$(\alpha_X+\beta+D+\lceil F^v\rceil+\lceil F^h\rceil)+dd^c\varphi_i\geq\lceil F^h\rceil+\lceil F_1^v\rceil \qquad \text{on } f^{-1}(U_i).$$

Then

$$(\alpha_X + \beta + D + \lceil F_2^v \rceil) + dd^c \varphi_i \geq 0 \qquad \text{on } f^{-1}(U_i), \tag{27}$$

and the claim is proved. □

4. Weak subadjunction

4.1. Definition. [Bou04, Defn.2.2] *Let X be a compact Kähler manifold, and let α be a cohomology class on X. We say that α is a modified Kähler class if it contains a Kähler current T such that the generic Lelong number $\nu(T, D)$ is zero for every prime divisor $D \subset X$.*

By [Bou04, Prop.2.3] a cohomology class is modified Kähler if and only if there exists a modification $\mu : \tilde{X} \to X$ and a Kähler class $\tilde{\alpha}$ on $\tilde{X}$ such that $\mu_* \tilde{\alpha} = \alpha$. For our purpose we have to fix some more notation:

4.2. Definition. *Let X be a compact Kähler manifold, and let α be a modified Kähler class on X. A log-resolution of α is a bimeromorphic morphism $\mu : \tilde{X} \to X$ from a compact Kähler manifold $\tilde{X}$ such that the exceptional locus is a simple normal crossings divisor $\sum_{j=1}^k E_j$ and there exists a Kähler class $\tilde{\alpha}$ on $\tilde{X}$ such that $\mu_* \tilde{\alpha} = \alpha$.*

The definition can easily be extended to arbitrary big classes by using the Boucksom's Zariski decomposition [Bou04, Thm.3.12].

4.3. Remark. If $\mu : \tilde{X} \to X$ is a log-resolution of α one can write

$$\mu^* \alpha = \tilde{\alpha} + \sum_{j=1}^k r_j E_j.$$

Applying the negativity lemma [KM98, Lemma 3.39] [BCHM10, 3.6.2] we see that $r_j > 0$ for all $j \in \{1, \ldots, k\}$.

4.4. Definition. *Let X be a compact Kähler manifold, and let α be a modified Kähler class on X. A subvariety $Z \subset X$ is a maximal lc centre if there exists a log-resolution $\mu : \tilde{X} \to X$ of α with exceptional locus $\sum_{j=1}^k E_j$ such that the following holds:*

- *Z is an irreducible component of $\mu(\operatorname{Supp} \sum_{j=1}^k E_j)$;*
- *if we write*

$$K_{\tilde{X}} + \tilde{\alpha} = \mu^*(K_X + \alpha) + \sum_{j=1}^k d_j E_j,$$

then $d_j \geq -1$ for every E_j mapping onto Z and (up to renumbering) we have $\mu(E_1) = Z$ and $d_1 = -1$.

Following the terminology for singularities of pairs we call the coefficients d_j the discrepancies of (X, α). Note that this terminology is somewhat abusive since d_j is not determined by the class α but depends on the choice of $\tilde{\alpha}$ (hence implicitly on the choice of a Kähler current T in α that is used to construct the log-resolution). Similarly it would be more appropriate to define Z as an lc centre of the pair (X, T) with $[T] \in \alpha$. Since most of the time we will only work with the cohomology class we have chosen to use this more convenient terminology.

We can now prove the weak subadjunction formula:

Proof of Theorem 1.5. Step 1. Geometric setup. Since $Z \subset X$ is a maximal lc centre of (X, α) there exists a log-resolution $\mu : \tilde{X} \to X$ of α with exceptional locus $\sum_{j=1}^{k} E_j$ such that Z is an irreducible component of $\mu(\operatorname{Supp} \sum_{j=1}^{k} E_j)$ and

$$K_{\tilde{X}} + \tilde{\alpha} = \mu^*(K_X + \alpha) + \sum_{j=1}^{k} d_j E_j, \tag{28}$$

satisfies $d_j \geq -1$ for every E_j mapping onto Z and (up to renumbering) we have $\mu(E_1) = Z$ and $d_1 = -1$. Let $\pi : X' \to X$ be an embedded resolution of Z, then (up to blowing up further $\tilde{X}$) we can suppose that there exists a factorisation $\psi : \tilde{X} \to X'$. Let $Z' \subset X'$ be the strict transform of Z. Since π is an isomorphism in the generic point of Z', the divisors E_j mapping onto Z' via ψ are exactly those mapping onto Z via μ. Denote by $Q_l \subset Z'$ the prime divisors that are images of divisors $E_1 \cap E_j$ via $\psi|_{E_1}$. Then we can suppose (up to blowing up further $\tilde{X}$) that the divisor

$$\sum_l (\psi|_{E_1})^* Q_l + \sum_{j=2}^{k} E_1 \cap E_j$$

has a support with simple normal crossings. We set

$$f := \psi|_{E_1}, \qquad \text{and} \qquad D = -\sum_{j=2}^{k} d_j D_j$$

where $D_j := E_j \cap E_1$. Note also that the desingularisation $\pi|_{Z'}$ factors through the normalisation $\nu : \tilde{Z} \to Z$, so we have a bimeromorphic morphism $\tau : Z' \to \tilde{Z}$ such that $\pi|_{Z'} = \nu \circ \tau$. We summarise the construction in a commutative diagram:

$$\begin{tikzcd}
 & & E_1 \arrow[dll, "f:=\psi|_{E_1}"'] \arrow[d, hook] \arrow[drr] & & \\
 & & \tilde{X} \arrow[dl, "\psi"'] \arrow[dr, "\mu"] & & \\
Z' \arrow[r, hook] \arrow[drr, "\tau"'] & X' \arrow[rr, "\pi"] & & X & Z \arrow[l, hook'] \\
 & & \tilde{Z} \arrow[urr, "\nu"'] & &
\end{tikzcd}$$

A priori there might be more than one divisor with discrepancy -1 mapping onto Z, but we can use the tie-breaking technique which is well-known in the context of singularities of pairs: recall that the class $\tilde{\alpha}$ is Kähler which is an open property. Thus we can choose $0 < \varepsilon_j \ll 1$ for all $j \in \{2, \ldots, k\}$ such that the class $\tilde{\alpha} + \sum_{j=2}^{k} \varepsilon_j E_j$ is Kähler. The decomposition

$$K_{\tilde{X}} + (\tilde{\alpha} + \sum_{j=2}^{k} \varepsilon_j E_j) = \mu^*(K_X + \alpha) - E_1 + \sum_{j=2}^{k} (d_j + \varepsilon_j) E_j$$

still satisfies the properties in Definition 4.4 and E_1 is now the unique divisor with discrepancy -1 mapping onto Z. Note that up to perturbing ε_j we can suppose that $d_j + \varepsilon_j$ is rational for every $j \in \{1, \ldots, k\}$. In order to simplify the notation we will suppose without loss of generality, that these properties already holds for the decomposition (28).

Outline of the strategy. The geometric setup above is analogous to the proof of Kawamata's subadjunction formula [Kaw98, Thm.1] and as in Kawamata's proof our aim is now to apply the positivity theorem 3.3 to f to relate $K_{Z'}$ and $(\pi|_{Z'})^*(K_X+\alpha)|_Z$. However since we deal with an lc centre that is not minimal we encounter some additional problems: the pair (E_1, D) is not necessarily (sub-)klt and the centre Z might not be regular in codimension one. In the end this will not change the relation between $K_{Z'}$ and $(\pi|_{Z'})^*(K_X+\alpha)|_Z$, but it leads to some technical computations which will be carried out in the Steps 3 and 4.

Step 2. Relative vanishing. Note that the $\mathbb{Q}$-divisor $-K_{\tilde{X}} - E_1 + \sum_{j=2}^k d_j E_j$ is μ-ample since its class is equal to $\tilde{\alpha}$ on the μ-fibres. Thus we can apply the relative Kawamata-Viehweg theorem (in its analytic version [Anc87, Thm.2.3] [Nak87]) to obtain that

$$R^1\mu_*\mathcal{O}_{\tilde{X}}(-E_1 + \sum_{j=2}^k \lceil d_j \rceil E_j) = 0.$$

Pushing the exact sequence

$$0 \to \mathcal{O}_{\tilde{X}}(-E_1 + \sum_{j=2}^k \lceil d_j \rceil E_j) \to \mathcal{O}_{\tilde{X}}(\sum_{j=2}^k \lceil d_j \rceil E_j) \to \mathcal{O}_{E_1}(\lceil -D \rceil) \to 0$$

down to X, the vanishing of R^1 yields a surjective map

$$\mu_*(\mathcal{O}_{\tilde{X}}(\sum_{j=2}^k \lceil d_j \rceil E_j)) \to (\mu|_{E_1})_*(\mathcal{O}_{E_1}(\lceil -D \rceil)). \tag{29}$$

Since all the divisors E_j are μ-exceptional, we see that $\mu_*(\mathcal{O}_{\tilde{X}}(\sum_{j=2}^k \lceil d_j \rceil E_j))$ is an ideal sheaf $\mathcal{I}$. Moreover, since $d_j > -1$ for all E_j mapping onto Z the sheaf $\mathcal{I}$ is isomorphic to the structure sheaf in the generic point of Z . In particular $(\mu|_{E_1})_*(\mathcal{O}_{E_1}(\lceil -D \rceil))$ has rank one.

Step 3. Application of the positivity result. By the adjunction formula we have

$$K_{E_1} + \tilde{\alpha}|_{E_1} - \sum_{j=2}^k d_j (E_j \cap E_1) = (\psi|_{E_1})^*(\pi|_{Z'})^*(K_X+\alpha)|_Z. \tag{30}$$

Since $\psi|_{E_1}$ coincides with $\mu|_{E_1}$ over the generic point of Z', we know by Step 2 that the direct image sheaf $f_*(\mathcal{O}_{E_1}(\lceil -D \rceil))$ has rank one. In particular f has connected fibres.

In general the boundary D does not satisfy the conditions a) and b) in Theorem 3.3, however we can still obtain some important information by applying Theorem 3.3 for a slightly modified boundary. but we can nevertheless obtain some important information by modifying the boundary D: note first that the fibration f is equidimensional over the complement of a codimension two set. In particular the direct image sheaf $f_*(\mathcal{O}_{E_1}(\lceil -D \rceil))$ is reflexive [Har80, Cor.1.7], hence locally free, on the complement of a codimension two set. Thus we can consider the first Chern class $c_1(f_*(\mathcal{O}_{E_1}(\lceil -D \rceil)))$ (cf. Definition 2.2). Set

$$L := (\pi|_{Z'})^*(K_X+\alpha)|_Z - K_{Z'},$$

then we claim that

$$(L + c_1(f_*(\mathcal{O}_{E_1}(\lceil -D \rceil)))) \cdot \omega'_1 \cdot \ldots \cdot \omega'_{\dim Z-1} \geq 0 \tag{31}$$

for any collection of nef classes ω'_j on Z'.

Proof of the claim. In the complement of a codimension two $B \subset Z'$ set the fibration $f|_{f^{-1}(Z' \setminus B)}$ is equidimensional, so the direct image sheaf $\mathcal{O}_{E_1}(\lceil -D^v \rceil)$ is reflexive. Since it has rank one we thus can write

$$f_*(\mathcal{O}_{E_1}(\lceil -D^v \rceil)) \otimes \mathcal{O}_{Z' \setminus B} = \mathcal{O}_{Z' \setminus B}(\sum e_l Q_l)$$

where $e_l \in \mathbb{Z}$ and $Q_l \subset Z'$ are the prime divisors introduced in the geometric setup. If $e_l > 0$ then e_l is the largest integer such that

$$(f|_{f^{-1}(Z' \setminus B)})^*(e_l Q_l) \subset \lceil -D^v \rceil.$$

In particular if D_j maps onto Q_l, then $d_j > -1$. If $e_l < 0$ there exists a divisor D_j that maps onto Q_l such that $d_j \leq -1$. Moreover if w_j is the coefficient of D_j in the pull-back $(f|_{f^{-1}(Z' \setminus B)})^* Q_l$, then e_l is the largest integer such that $d_j - e_l w_j > -1$ for every divisor D_j mapping onto Q_l. Thus if we set

$$\tilde{D} := D + \sum e_l f^* Q_l,$$

then $\tilde{D}$ has normal crossings support (cf. Step 1) and satisfies the condition a) in Theorem 3.3. Moreover if we denote by $\tilde{D} = \tilde{D}^h + \tilde{D}^v$ the decomposition in horizontal and vertical part, then $\tilde{D}^h = D^h$ and $\tilde{D}^v = D^v + \sum e_l f^* Q_l$. Since we did not change the horizontal part, the direct image $f_*(\mathcal{O}_{E_1}(\lceil -\tilde{D} \rceil))$ has rank one. Since $\sum e_l f^* Q_l$ has integral coefficients, the projection formula shows that

$$(f_*(\mathcal{O}_{E_1}(\lceil -\tilde{D}^v \rceil)))^{**} \simeq (f_*(\mathcal{O}_{E_1}(\lceil -D^v \rceil)))^{**} \otimes \mathcal{O}_{Z'}(-\sum e_l Q_l) \simeq \mathcal{O}_{Z'}.$$

Thus we satisfy the condition b) in Theorem 3.3. Finally note that

$$K_{E_1/Z} + \tilde{\alpha}|_{E_1} + \tilde{D} = f^*(L + \sum e_l Q_l).$$

So if we set $\tilde{L} := L + \sum e_l Q_l$, then

$$\tilde{L} + c_1(f_*(\mathcal{O}_{E_1}(\lceil -\tilde{D} \rceil))) = L + c_1(f_*(\mathcal{O}_{E_1}(\lceil -D \rceil))). \tag{32}$$

Now we apply Theorem 3.3 and obtain

$$\tilde{L} \cdot \omega'_1 \cdot \ldots \cdot \omega'_{\dim Z'-1} \geq 0.$$

Yet by the conditions a) and b) there exists an ideal sheaf $\mathcal{I}$ on Z' that has cosupport of codimension at least two and $f_*(\mathcal{O}_{E_1}(\lceil -\tilde{D} \rceil)) \simeq \mathcal{I} \otimes \mathcal{O}_{Z'}(B)$ with B an effective divisor on Z'. Thus $c_1(f_*(\mathcal{O}_{E_1}(\lceil -\tilde{D} \rceil)))$ is represented by the effective divisor B and the claim follows from (32).

Step 4. Final computation. In view of our definition of the intersection product on $\tilde{Z}$ (cf. Definition 2.7) we are done if we prove that

$$L \cdot \tau^* \omega_1 \cdot \ldots \cdot \tau^* \omega_{\dim Z-1} \geq 0$$

where the ω_j are the nef cohomology classes from the statement of Theorem 1.5. We claim that

$$c_1(f_*(\mathcal{O}_{E_1}(\lceil -D \rceil))) = -\Delta_1 + \Delta_2 \tag{33}$$

where Δ_1 is an effective divisor and Δ_2 is a divisor such that $\pi|_{Z'}(\operatorname{Supp}\Delta_2)$ has codimension at least two in Z. Assuming this claim for the time being let us see how to conclude: by (31) we have

$$(L + c_1(f_*(\mathcal{O}_{E_1}(\lceil -D \rceil)))) \cdot \tau^*\omega_1 \cdot \ldots \cdot \tau^*\omega_{\dim Z-1} \geq 0. \tag{34}$$

Since the normalisation ν is finite and $\pi|_{Z'}(\operatorname{Supp}\Delta_2)$ has codimension at least two in Z, we see that $\tau(\operatorname{Supp}\Delta_2)$ has codimension at least two in $\tilde{Z}$. Thus we have

$$c_1(f_*(\mathcal{O}_{E_1}(\lceil -D \rceil))) \cdot \tau^*\omega_1 \cdot \ldots \cdot \tau^*\omega_{\dim Z-1} = -\Delta_1 \cdot \tau^*\omega_1 \cdot \ldots \cdot \tau^*\omega_{\dim Z-1} \leq 0.$$

Hence the statement follows from (34).

Proof of the claim. Applying as in Step 2 the relative Kawamata-Viehweg vanishing theorem to the morphism ψ we obtain a surjection

$$\psi_*(\mathcal{O}_{\tilde{X}}(\sum_{j=2}^{k} \lceil d_j \rceil E_j)) \to (\psi|_{E_1})_*(\mathcal{O}_{E_1}(\lceil -D \rceil))$$

In order to verify (33) note first that some of the divisors E_j might not be ψ-exceptional, so it is not clear if $\psi_*(\mathcal{O}_{\tilde{X}}(\sum_{j=2}^{k} \lceil d_j \rceil E_j))$ is an ideal sheaf. However if we restrict the surjection (29) to Z we obtain a surjective map

$$\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_Z \to (\pi|_{Z'})_*(f_*(\mathcal{O}_{E_1}(\lceil -D \rceil))), \tag{35}$$

where $\mathcal{I}$ is the ideal sheaf introduced in Step 2. There exists an analytic set $B \subset Z$ of codimension at least two such that

$$Z' \setminus \pi^{-1}(B) \to Z \setminus B$$

is isomorphic to the normalisation of $Z \setminus B$. In particular the restriction of π to $Z' \setminus \pi^{-1}(B)$ is finite, so the natural map

$$(\pi|_{Z'})^*(\pi|_{Z'})_*(f_*(\mathcal{O}_{E_1}(\lceil -D \rceil))) \to f_*(\mathcal{O}_{E_1}(\lceil -D \rceil))$$

is surjective on $Z' \setminus \pi^{-1}(B)$. Pulling back is right exact, so composing with the surjective map (35) we obtain a map from an ideal sheaf to $f_*(\mathcal{O}_{E_1}(\lceil -D \rceil))$ that is surjective on $Z' \setminus \pi^{-1}(B)$. Thus $c_1(f_*(\mathcal{O}_{E_1}(\lceil -D \rceil)))$ decomposes into an antieffective divisor $-\Delta_1$ mapping into the non-normal locus of $Z \setminus B$ and a divisor Δ_2 mapping into B. Since B has codimension at least two this proves the claim. □

4.5. Remark. In Step 3 of the proof of Theorem 1.5 above we introduce a "boundary" $c_1(f_*(\mathcal{O}_M(\lceil -D \rceil)))$ so that we can apply Theorem 3.3. One should note that this divisor is fundamentally different from the divisor Δ appearing in [Kaw98, Thm.1, Thm.2]. In fact for a minimal lc centre Kawamata's arguments show that $c_1(f_*(\mathcal{O}_M(\lceil -D \rceil))) = 0$, his boundary divisor Δ is defined in order to obtain the stronger result that $L - \Delta$ is nef. We have to introduce $c_1(f_*(\mathcal{O}_M(\lceil -D \rceil)))$ since we want to deal with non-minimal centres.

5. Positivity of relative adjoint classes, part 2

Convention : In this section, we use the following convention. Let U be a open set and $(f_m)_{m\in\mathbb{N}}$ be a sequence of smooth functions on U. We say that

$$\|f_m\|_{C^\infty(U)} \to 0,$$

if for every open subset $V \Subset U$ and every index α, we have

$$\|\partial^\alpha f_m\|_{C^0(V)} \to 0.$$

Similarly, in the case $(f_m)_{m\in\mathbb{N}}$ are smooth formes, we say that $\|f_m\|_{C^\infty(U)} \to 0$ if every component tends to 0 in the above sense.

Before giving the main theorem of this section, we need two preparatory lemmas. The first comes from [LAE02, Part II, Thm 1.3] :

5.1. Lemma.[LAE02, Part II, Thm 1.3] *Let X be a compact Kähler manifold and let α be a closed smooth real 2-form on X. Then we can find a strictly increasing sequence of integers $(s_m)_{m\geq 1}$ and a sequence of hermitian line bundles (not necessary holomorphic) $(F_m, D_{F_m}, h_{F_m})_{m\geq 1}$ on X such that*

$$\lim_{m\to+\infty} \|\frac{\sqrt{-1}}{2\pi}\Theta_{h_{F_m}}(F_m) - s_m\alpha\|_{C^\infty(X)} = 0. \tag{36}$$

Here D_{F_m} is a hermitian connection with respect to the smooth hermitian metric h_{F_m} and $\Theta_{h_{F_m}}(F_m) = D_{F_m}\circ D_{F_m}$.
Moreover, let (W_j) be a small Stein cover of X and let $e_{F_m,j}$ be a basis of an isometric trivialisation of F_m over W_j i.e., $\|e_{F_m,j}\|_{h_m} = 1$. Then we can ask the hermitian connections D_{F_m} to satisfy the following additional condition: for the $(0,1)$-part of D_{F_m} on W_j : $D''_{F_m} = \overline{\partial} + \beta^{0,1}_{m,j}$, we have

$$\|\frac{1}{s_m}\beta^{0,1}_{m,j}\|_{C^\infty(W_j)} \leq C\|\alpha\|_{C^\infty(X)}, \tag{37}$$

where C is a uniform constant independent of m and j.

Proof. Thanks to [LAE02, Part II, Thm 1.3], we can find a strictly increasing integer sequence $(s_m)_{m\geq 1}$ and closed smooth 2-forms $(\alpha_m)_{m\geq 1}$ on X, such that

$$\lim_{m\to+\infty} \|\alpha_m - s_m\alpha\|_{C^\infty(X)} = 0 \qquad \text{and} \qquad \alpha_m \in H^2(X,\mathbb{Z}).$$

Since (W_j) are small Stein open sets, we can find some smooth 1-forms $\beta_{m,j}$ on W_j such that

$$\frac{1}{2\pi}\cdot d\beta_{m,j} = \alpha_m \text{ on } W_j \qquad \text{and} \qquad \|\frac{1}{s_m}\beta_{m,j}\|_{C^\infty(W_j)} \leq C\|\alpha\|_{C^\infty(X)} \tag{38}$$

for a constant C independent of m and j.
By using the standard construction (cf. for example [Dem, V, Thm 9.5]), the form $(\beta_{m,j})_j$ induces a hermitian line bundle (F_m, D_m, h_{F_m}) on X such that $D_m = d + \frac{\sqrt{-1}}{2\pi}\beta_{m,j}$ with respect to an isometric trivialisation over W_j. Then

$$\|\frac{\sqrt{-1}}{2\pi}\Theta_{h_{F_m}}(F_m) - s_m\alpha\|_{C^\infty(X)} = \|\alpha_m - s_m\alpha\|_{C^\infty(X)} \to 0.$$

Let $\beta^{0,1}_{m,j}$ be the $(0,1)$-part of $\beta_{m,j}$. Then (38) implies (37). □

We now recall a $L^{\frac{2}{m}}$-version Ohsawa-Takegoshi extension theorem proved in [BP10, Prop 0.2]

5.2. Proposition.[BP10, Prop 0.2] *Let $\Omega \subset \mathbb{C}^n$ be a ball of radius r and let $h : \Omega \to \mathbb{C}$ be a holomorphic function such that $\sup_\Omega |h| \leq 1$. Moreover, we assume that the gradient ∂h of h is nowhere zero on the set $V := (h = 0)$. Let φ be a plurisubharmonic function such that its restriction to V is well-defined. Let $d\lambda$ be* *the standard volume form on $\mathbb{C}^n$ and $d\lambda_V$ be its restriction on V. Then for any holomorphic function $f : V \to \mathbb{C}$ and any $m \in \mathbb{N}$ such that*

$$\int_V |f|^{\frac{2}{m}} e^{-\varphi} \frac{d\lambda_V}{|\partial h|^2} \leq 1,$$

there exists a function $F \in \mathcal{O}(\Omega)$ such that :

(i) $F|_V = f$

(ii) *The following $L^{\frac{2}{m}}$ bound holds*

$$\int_\Omega |F|^{\frac{2}{m}} e^{-\varphi} d\lambda \leq C_0 \int_V |f|^{\frac{2}{m}} e^{-\varphi} \frac{d\lambda_V}{|\partial h|^2},$$

where C_0 is an absolute constant as in the standard Ohsawa-Takegoshi theorem.

We need here a slightly global version of the above proposition :

5.3. Lemma. *Let $p : X \to Y$ be a fibration between two compact Kähler manifolds. Fix a Kähler metric ω_X (resp. ω_Y) on X (resp. Y). Let h be the metric on $K_{X/Y}$ induced by ω_X and ω_Y. Let U be a small Stein open set in X and let $m \in \mathbb{N}$. Let φ be a quasi-psh function on U such that*

$$dd^c\varphi - m\frac{\sqrt{-1}}{2\pi}\Theta_h(K_{X/Y}) \geq 0.$$

Let y be a point in Y such that X_y is a smooth fiber. Then for any holomorphic function $f : U \cap X_y \to \mathbb{C}$, we can find a function $F \in \mathcal{O}(U)$ such that

(i) $F|_{U\cap X_y} = f$

(ii) *The next $L^{\frac{2}{m}}$ estimate holds*

$$\int_U |F|^{\frac{2}{m}} e^{-\frac{2\varphi}{m}} \omega_X^{\dim X} \leq C_0 \int_{U\cap X_y} |f|^{\frac{2}{m}} e^{-\frac{2\varphi}{m}} \omega_X^{\dim X} / p^*\omega_Y^{\dim Y},$$

where C_0 is a uniform constant independent of f, m and φ.

Moreover, set $M_\varphi := \frac{1}{m}\sup_{x,y\in U}|\varphi(x) - \varphi(y)|$. Let $V \Subset U$. Then there exists a constant C_1 which depends only on V and U, such that for every $y \in V$ and $f \in H^0(U \cap X_y, \mathcal{O}_X)$ we have

$$e^{-\frac{2}{m}\varphi(x)}|f|^{\frac{2}{m}}(x) \leq C_1 e^{2M_\varphi} \int_{U\cap X_y} |f|^{\frac{2}{m}} e^{-\frac{2\varphi}{m}} \omega_X^{\dim X} / p^*\omega_Y^{\dim Y} \qquad \text{for every } x \in X_y \cap V.$$

Proof. For the first part, let $L_m = \mathcal{O}_X - mK_{X/Y}$ with the metric $e^{-2\varphi}h^{-m}$ on U. Then its curvature:

$$\frac{\sqrt{-1}}{2\pi}\Theta(L_m) = dd^c\varphi - m\frac{\sqrt{-1}}{2\pi}\Theta_h(K_{X/Y}) \geq 0.$$

By applying the same proof in [BP10, A.1] to $\mathcal{O}_X = mK_{X/Y} + L_m$ over U, we can find a function $F \in \mathcal{O}(U)$ satisfying both (i) and (ii).

For the second part, note first that from the definition of M_φ we have

$$e^{-\frac{2}{m}\varphi(x)} \cdot \int_U |F|^{\frac{2}{m}} \omega_X^{\dim X} \leq e^{2M_\varphi} \int_U |F|^{\frac{2}{m}} e^{-2\frac{\varphi}{m}} \omega_X^{\dim X}.$$

Thanks to (ii) we thus obtain

$$(39)\qquad e^{-\frac{2}{m}\varphi(x)}\cdot\int_U |F|^{\frac{2}{m}}\omega_X^{\dim X}\le C_0 e^{2M_\varphi}\int_{U\cap X_y}|f|^{\frac{2}{m}}e^{-\frac{\varphi}{m}}\omega_X^{\dim X}/p^*\omega_Y^{\dim Y}.$$

Since $V\Subset U$, we can find a real number $r>0$ such that for every $x\in V$, its r-neighbourhood $B_r(x)$ is contained in U. Since $|F|^{\frac{2}{m}}$ is psh on U, by mean value inequality, for every $x\in V$, we have

$$(40)\qquad e^{-\frac{2}{m}\varphi(x)}\cdot|f|^{\frac{2}{m}}(x)\le\frac{1}{r^{2\dim X}}e^{-\frac{2}{m}\varphi(x)}\cdot\int_U|F|^{\frac{2}{m}}\omega_X^{\dim X}.$$

The lemma is thus proved by combining (39) and (40). □

Now we can prove the main theorem of this section.

5.4. Theorem. *Let X and Y be two compact Kähler manifolds and let $f: X\to Y$ be a surjective map with connected fibres such that the general fibre F is simply connected and*

$$H^0(F,\Omega_F^2)=0.$$

Let ω be a Kähler form on X such that $c_1(K_F)+[\omega|_F]$ is a pseudoeffective class. Then $c_1(K_{X/Y})+[\omega]$ is pseudoeffective.

Proof. Being pseudoeffective is a closed property, so we can assume without loss of generality that $c_1(K_F)+[\omega|_F]$ is big on F.

Step 1: Preparation, Stein Cover.

Fix two Kähler metrics ω_X, ω_Y on X and Y respectively. Let h be the smooth hermitian metric on $K_{X/Y}$ induced by ω_X and ω_Y. Set $\alpha:=\frac{\sqrt{-1}}{2\pi}\Theta_h(K_{X/Y})$. Thanks to Lemma 5.1, there exist a strictly increasing sequence of integers $(s_m)_{m\ge1}$ and a sequence of hermitian line bundles (not necessary holomorphic) $(F_m, D_{F_m}, h_{F_m})_{m\ge1}$ on X such that

$$(41)\qquad \|\frac{\sqrt{-1}}{2\pi}\Theta_{h_{F_m}}(F_m)-s_m(\alpha+\omega)\|_{C^\infty(X)}\to 0.$$

By our assumption on F we can find a non empty Zariski open subset Y_0 of Y such that f is smooth over Y_0 and $R^if_*\mathcal{O}_X=0$ on Y_0 for every $i=1,2$. Let $(U_i)_{i\in I}$ be a Stein cover of Y_0. Therefore

$$(42)\qquad H^{0,2}(f^{-1}(U_i),\mathbb{R})=0\qquad\text{for every } i\in I.$$

Step 2: Construction of the metric.

We would like to construct in this step a relative Bergman kernel type quasi-psh function φ_i on $f^{-1}(U_i)$, such that

$$(43)\qquad \alpha+\omega+dd^c\varphi_i\ge 0\qquad\text{on } f^{-1}(U_i)$$

in the sense of currents.

In fact, thanks to (42), we know that the $(0,2)$-part of $\Theta_{h_{F_m}}(F_m)$ is $\overline{\partial}$-exact on $f^{-1}(U_i)$. Combining this with (41), we can find holomorphic line bundles $L_{i,m}$ on $f^{-1}(U_i)$ equipped with smooth hermitian metrics $h_{i,m}$ such that

$$(44)\qquad \|\frac{\sqrt{-1}}{2\pi}\Theta_{h_{F_m}}(F_m)-\frac{\sqrt{-1}}{2\pi}\Theta_{h_{i,m}}(L_{i,m})\|_{C^\infty(f^{-1}(U_i))}\to 0.$$

By construction, we have

$$\frac{\sqrt{-1}}{2\pi}\Theta_{h_{i,m}}(L_{i,m}) - s_m\frac{\sqrt{-1}}{2\pi}\Theta_h(K_{X/Y}) = \frac{\sqrt{-1}}{2\pi}\Theta_{h_{i,m}}(L_{i,m}) - s_m\alpha$$

$$= (\frac{\sqrt{-1}}{2\pi}\Theta_{h_{i,m}}(L_{i,m}) - \frac{\sqrt{-1}}{2\pi}\Theta_{h_{F_m}}(F_m)) + (\frac{\sqrt{-1}}{2\pi}\Theta_{h_{F_m}}(F_m) - s_m(\alpha+\omega)) + s_m\omega.$$

Thanks to the estimates (41) and (44), the first two terms of the right-hand side of the above equality tends to 0. Therefore we can find a sequence of open sets $U_{i,m} \Subset U_i$, such that $\cup_{m\geq 1} U_{i,m} = U_i$ and for every j one has $U_{i,m} \Subset U_{i,m+1}$, and

$$\frac{\sqrt{-1}}{2\pi}\Theta_{h_{i,m}}(L_{i,m}) - s_m\frac{\sqrt{-1}}{2\pi}\Theta_h(K_{X/Y}) \geq 0 \qquad \text{on } f^{-1}(U_{i,m}). \tag{45}$$

Let $\varphi_{i,m}$ be the s_m-Bergman kernel associated to the pair (cf. Remark 3.2)

$$(L_{i,m} = s_m K_{X/Y} + (L_{i,m} - s_m K_{X/Y}), h_{i,m}) \tag{46}$$

i.e., $\varphi_{i,m}(x) := \sup\limits_{g\in A} \frac{1}{s_m}\ln|g|_{h_{i,m}}(x)$, where

$$A := \{g \mid g \in H^0(X_{f(x)}, L_{i,m}), \int_{X_{f(x)}} |g|^{\frac{2}{s_m}}_{h_{i,m}}\omega_X^{\dim X}/f^*\omega_Y^{\dim Y} = 1\}.$$

Thanks to (45), we can apply Theorem 3.1 to the pair (46) over $f^{-1}(U_{i,m})$. In particular, we have

$$(\alpha+\omega) + dd^c\varphi_{i,m} \geq 0 \qquad \text{on } f^{-1}(U_{i,m}). \tag{47}$$

By Ohsawa-Takegoshi extension theorem (for example [BP10, Remark 2.3]), $\sup\limits_{m\geq k}\varphi_{i,m}$ is still a function. Let φ_i be the regularization of $\lim\limits_{k\to+\infty}\sup\limits_{m\geq k}\varphi_{i,m}$. By monotone convergence theorem, φ_i is still quasi-psh. As $\cup_{m\geq 1}U_{i,m} = U_i$, (47) implies thus that

$$\alpha + \omega + dd^c\varphi_i \geq 0 \qquad \text{on } f^{-1}(U_i).$$

Step 3: Extension, final conclusion.

We claim that

Claim 1. $\varphi_i = \varphi_j$ on $f^{-1}(U_i \cap U_j)$ for every i, j.

Claim 2. For every small Stein open set V in X, we can find a constant C_V depending only on V such that

$$\varphi_i(x) \leq C_V \qquad \text{for every } i \text{ and } x \in V \cap f^{-1}(U_i).$$

We postpone the proof of these two claims and finish first the proof of the theorem. Thanks to Claim 1, $(\varphi_i)_{i\in I}$ defines a global quasi-psh function φ on $f^{-1}(Y_0)$. Then (43) implies that

$$\alpha + \omega + dd^c\varphi \geq 0 \qquad \text{on } f^{-1}(Y_0).$$

Thanks to Claim 2, we have $\varphi \leq C_V$ on $V \cap f^{-1}(Y_0)$. Therefore φ can be extended as a quasi-psh function on V. Since Claim 2 is true for every small Stein open set V, φ can be extended as a quasi-psh function on X and satisfies

$$\alpha + \omega + dd^c\varphi \geq 0 \qquad \text{on } X.$$

As a consequence, $c_1(K_{X/Y}) + [\omega]$ is pseudoeffective. □

We are left to prove the two claims in the proof of the theorem.

Proof of Claim 1. Let $y \in U_i \cap U_j$ be a generic point. Thanks to (44), we have

$$\lim_{m\to+\infty} \| \frac{\sqrt{-1}}{2\pi}\Theta_{h_{i,m}}(L_{i,m})|_{X_y} - \frac{\sqrt{-1}}{2\pi}\Theta_{h_{j,m}}(L_{j,m})|_{X_y} \|_{C^\infty(X_y)} = 0. \tag{48}$$

When m is large enough, (48) implies that

$$c_1(L_{i,m}|_{X_y}) = c_1(L_{j,m}|_{X_y}) \in H^{1,1}(X_y) \cap H^2(X_y, \mathbb{Z}).$$

Combining this with the fact that X_y is simply connected, we have

$$L_{i,m}|_{X_y} = L_{j,m}|_{X_y} \qquad \text{for } m \gg 1. \tag{49}$$

Under the isomorphism of (49), by applying $\partial\overline{\partial}$-lemma, (48) imply the existence of constants $c_m \in \mathbb{R}$ and smooth functions $\tau_m \in C^\infty(X_y)$ such that

$$h_{i,m} = h_{j,m} e^{c_m+\tau_m} \text{ on } X_y \qquad \text{and} \qquad \lim_{m\to+\infty} \|\tau_m\|_{C^\infty(X_y)} = 0.$$

Combining with the construction of $\varphi_{i,m}$ and $\varphi_{j,m}$, we know that

$$\|\varphi_{i,m} - \varphi_{j,m}\|_{C^0(X_y)} \leq \|\tau_m\|_{C^0(X_y)} \to 0.$$

Therefore

$$\varphi_i|_{X_y} = \varphi_j|_{X_y} \tag{50}$$

As (50) is proved for every generic point $y \in U_i \cap U_j$, we have

$$\varphi_i = \varphi_j \qquad \text{on } f^{-1}(U_i \cap U_j).$$

The claim is proved. □

It remains to prove the claim 2. Note that $(L_{i,m}, h_{i,m})$ is defined only on $f^{-1}(U_i)$, we can not directly apply Lemma 5.3 to $(L_{i,m}, h_{i,m})$. Although the proof of the claim 2 is some complicated, the idea is very simple: Thanks to the construction of F_m and $L_{i,m}$, by using $\partial\overline{\partial}$-lemma, we can prove that, after multiplying by a constant (which depends on $f(x) \in Y$), the difference between $h_{F_m}|_{X_{f(x)}}$ and $h_{i,m}|_{X_{f(x)}}$ is uniformly controlled for $m \gg 1$ [5]. Therefore $(F_m|_{X_{f(x)}}, h_{F_m})$ is not far from $(L_{i,m}|_{X_{f(x)}}, h_{i,m})$. Note that, using again (41), $F_m|_V$ is not far from a holomorphic line bundle over V. Combining Lemma 5.3 with these two facts, we can finally prove the claim 2.

Proof of Claim 2. *Step 1: Global approximation.*
Fix a small Stein cover $(W_j)_{j=1}^N$ of X. Without loss of generality, we can assume that $V \Subset W_1$. Let $(F_m, D_{F_m}, h_{F_m})_{m\geq 1}$ be the hermitian line bundles (not necessary holomorphic) constructed in the step 1 of the proof of Theorem 5.4. Let $e_{F_m,j}$ be a basis of a isometric trivialisation of F_m over W_j i.e., $\|e_{F_m,j}\|_{h_{F_m}} = 1$. Under this trivialisation, we suppose that the $(0,1)$-part of D_{F_m} on W_j is $D''_{F_m} = \overline{\partial} + \beta^{0,1}_{m,j}$, where $\beta^{0,1}_{m,j}$ is a smooth $(0,1)$-form on W_j. By Lemma 5.1, we can assume that

$$\|\frac{1}{s_m}\beta^{0,1}_{m,j}\|_{C^\infty(W_j)} \leq C_1\|\alpha+\omega\|_{C^\infty(X)} \tag{51}$$

for a uniform constant C_1 independent of m and j.

Step 2: Local estimation near V.

[5] The bigness of $m \gg 1$ depends on $f(x)$.

Thanks to (41), we know that F_m is not far from a holomorphic line bundle. In this step, we would like to give a more precise description of it in a neighbourhood of the Stein open set V.

Since W_1 is a small Stein open set, thanks to (41), there exists smooth functions $\{\psi_m\}_{m\geq 1}$ on W_1 and smooth $(0,1)$-forms $\{\sigma_m^{0,1}\}_{m\geq 1}$ on W_1 such that

(i) $(F_m, D''_{F,m} + \sigma_m^{0,1}) \simeq \mathcal{O}_{W_1}$ on W_1 for every $m \in \mathbb{N}$.
(ii) $\frac{\sqrt{-1}}{2\pi}\Theta_{h_{F_m}e^{-\psi_m}}(F_m) = s_m(\alpha + \omega)$ on W_1 for every $m \in \mathbb{N}$. [6]
(iii) $\lim\limits_{m\to+\infty}(\|\sigma_m^{0,1}\|_{C^\infty(W_1)} + \|\psi_m\|_{C^\infty(W_1)}) = 0$.

Thanks to (i) we have $(D''_{F_m} + \sigma_m^{0,1})^2 = 0$. Then $\beta_{m,1}^{0,1} + \sigma_m^{0,1}$ is $\overline{\partial}$-closed. Applying standard L^2-estimate, by restricting on some a little bit smaller open subset of W_1 (we still denote it by W_1 for simplicity), there exists a smooth function η_m on W_1 such that

$$\overline{\partial}\eta_m = \beta_{m,1}^{0,1} + \sigma_m^{0,1} \qquad \text{on } W_1 \tag{52}$$

and

$$\frac{1}{s_m}\|\eta_m\|_{C^\infty(W_1)} \leq \frac{C_2}{s_m}\|\beta_{m,1}^{0,1} + \sigma_m^{0,1}\|_{C^\infty(W_1)}$$

for a constant C_2 independent of m. Combining this with (51) and (iii), we get

$$\overline{\lim}_{m\to+\infty}\frac{1}{s_m}\|\eta_m\|_{C^\infty(W_1)} \leq C_1 \cdot C_2. \tag{53}$$

Moreover, by (52), $e^{-\eta_m} \cdot e_{F_m,1}$ is a holomorphic basis of $(W_1, F_m, D''_{F_m} + \sigma_m^{0,1})$.

Step 3: Fibrewise estimate.

Let $x \in V \cap f^{-1}(U_i)$ and set $y := f(x)$. In this step, we would like to compare h_{F_m} and $h_{i,m}$ on X_y.

By (41) and the rational connectedness of X_y, when m is large enough, we can find a smooth $(0,1)$-forms $\tau_m^{0,1}$ on X_y such that

$$\lim_{m\to+\infty}\|\tau_m^{0,1}\|_{C^\infty(X_y)} = 0 \qquad \text{and} \qquad (F_m, D''_{F_m} + \tau_m^{0,1})|_{X_y} \simeq L_{i,m}|_{X_y}. \tag{54}$$

Using again (41), we can thus find smooth functions $\widetilde{\psi}_m$ on X_y such that

$$\lim_{m\to+\infty}\|\widetilde{\psi}_m\|_{C^\infty(X_y)} = 0 \qquad \text{and} \qquad \Theta_{h_{F_m}e^{-\widetilde{\psi}_m}}(F_m)|_{X_y} = \Theta_{h_{i,m}}(L_{i,m})|_{X_y}. \tag{55}$$

Here the curvature $\Theta_{h_{F_m}e^{-\widetilde{\psi}_m}}(F_m)$ is calculated for the Chern connection with respect to $h_{F_m}e^{-\widetilde{\psi}_m}$ and $D''_{F_m} + \tau_m^{0,1}$, and $\Theta_{h_{i,m}}(L_{i,m})$ is the curvature for the holomorphic line bundle $L_{i,m}$ with respect to the metric $h_{i,m}$.

By using $\partial\overline{\partial}$-lemma over X_y, under the isomorphism of (54), (55) implies the existence of a constant $c_{m,y}$ such that

$$h_{F_m} \cdot e^{-\widetilde{\psi}_m} = h_{i,m} \cdot e^{-c_{m,y}} \qquad \text{on } X_y. \tag{56}$$

Here $c_{m,y}$ is a constant on X_y which depends only on m and y.

[6] Here $\Theta_{h_{F_m}e^{-\psi_m}}(F_m)$ is the curvature for the Chern connection on F_m with respect to complex structure $D''_{F,m} + \sigma_m^{0,1}$ and the metric $h_{F_m} \cdot e^{-\psi_m}$.

Like in Step 2, we want to find a holomorphic basis of $(F_m, D''_{F_m} + \tau_m^{0,1})$ over X_y. Thanks to (54), by the same reason as in Step 2, there exists some smooth functions ζ_m on $X_y \cap W_1$, such that

$$\overline{\partial}\zeta_m = \beta_{m,1}^{0,1} + \tau_m^{0,1} - \overline{\partial}\eta_m \qquad \text{on } X_y \cap W_1$$

and

$$\|\zeta_m\|_{C^\infty(X_y\cap W_1)} \leq C_y \|\beta_{m,1}^{0,1} + \tau_m^{0,1} - \overline{\partial}\eta_m\|_{C^\infty(X_y\cap W_1)} = C_y\|\tau_m^{0,1} - \sigma_m^{0,1}\|_{C^\infty(X_y\cap W_1)}$$

for a constant C_y independent of m, but depending on y.
Therefore $(e^{-\zeta_m-\eta_m}\cdot e_{F_m,j})|_{X_y\cap W_1}$ is holomorphic basis of $(X_y\cap W_1, F_m, D''_{F_m}+\tau_m^{0,1})$ and satisfies

$$(57) \qquad \lim_{m\to+\infty} \frac{1}{s_m}\|\zeta_m\|_{C^\infty(X_y\cap W_1)} \leq \lim_{m\to+\infty}\frac{C_y}{s_m}\|\tau_m^{0,1} - \sigma_m^{0,1}\|_{C^\infty(X_y\cap W_1)} = 0.$$

Step 4: Final conclusion.
Let x and y be the points chosen in the beginning of Step 3. To prove the claim, we need to estimate $\varphi_i(x)$. By the definition of $\varphi_{i,m}$, there exists a $g \in H^0(X_y, L_{i,m})$ such that

$$\varphi_{i,m}(x) = \frac{1}{s_m}\ln|g|_{h_{i,m}}(x) \qquad \text{and} \qquad \int_{X_y} |g|_{h_{i,m}}^{\frac{2}{s_m}} \omega_X^{\dim X}/\omega_Y^{\dim Y} = 1.$$

Using the isomorphism (54) and the metric estimations (55) and (56), we get the key point of the proof : there exists a $\widetilde{g} \in H^0(X_y, F_m, D''_{F_m} + \tau_m^{0,1})$[7] such that

$$(58) \qquad \int_{X_y} |\widetilde{g}|_{h_{F_m}}^{\frac{2}{s_m}} \omega_X^{\dim X}/\omega_Y^{\dim Y} = 1 \qquad \text{and} \qquad \varphi_{i,m}(x) \leq \frac{1}{s_m}\ln|\widetilde{g}|_{h_{F_m}}(x) + 1$$

where m is large enough. Here we use the important fact that $c_{m,y}$ is constant on X_y (although it might be very large). [8]
By our construction, we have $\widehat{g} := e^{\zeta_m}\cdot\widetilde{g} \in H^0(X_y\cap W_1, F_m, D''_{F_m} + \sigma_m^{0,1})$. Thanks to (57) and (58), when m is large enough, we have

$$(59) \qquad \int_{X_y\cap W_1} |\widehat{g}|_{h_{F_m}}^{\frac{2}{s_m}} \omega_X^{\dim X}/\omega_Y^{\dim Y} \leq 2$$

and

$$(60) \qquad \varphi_{i,m}(x) \leq \frac{1}{s_m}\ln|\widehat{g}|_{h_{F_m}}(x) + 2.$$

We now use Lemma 5.3 to estimate $\frac{1}{s_m}\ln|\widehat{g}|_{h_{F_m}}(x)$. Thanks to step 2, $e^{-\eta_m}e_{F_m,1}$ is a holomorphic basis of $(W_1, F_m, D''_m + \sigma_m^{0,1})$ and

$$(F_m|_{W_1}, D''_m + \sigma_m^{0,1}) \simeq \mathcal{O}_{W_1}.$$

Combining with (iii), we can thus apply Lemma 5.3 to the pair

$$(W_1, F_m, D''_m + \sigma_m^{0,1}, h_{F_m}e^{-\psi_m}).$$

In particular, by the taking the holomorphic trivialisation with respect to the holomorphic basis $e^{-\eta_m}e_{F_m,1}$, The constant M_φ in Lemma 5.3 is bounded by

[7] It means that $\widetilde{g}$ is a holomorphic section of F_m on X_y with respect to the complex structure $D''_{F_m} + \tau_m^{0,1}$.

[8] It is helpful to compare the argument here with (14).

$\frac{1}{s_m}(\|\psi_m\|_{C^0(W_1)} + \|\eta_m\|_{C^0(W_1)})$ in this situation. By using the estimates for ψ_m and η_m, we know that

$$\frac{1}{s_m}(\|\psi_m\|_{C^0(W_1)} + \|\eta_m\|_{C^0(W_1)}) \leq 2C_1 \cdot C_2$$

when m is large enough. By applying Lemma 5.3, (59) implies that

$$|\widehat{g}|^{\frac{2}{s_m}}_{h_{F_m}}(x) \leq e^{2C_1 \cdot C_2} \cdot C_{V,W_1},$$

where C_{V,W_1} is a uniformly constant depending only on V and W_1. Combining this with (60), we have

$$\varphi_{i,m}(x) \leq C_1 \cdot C_2 + \frac{\ln C_{V,W_1}}{2} + 2 \qquad \text{for } m \gg 1.$$

Therefore

$$\varphi_i(x) \leq C_1 \cdot C_2 + \frac{\ln C_{V,W_1}}{2} + 2.$$

Since the constants C_1, C_2 and C_{V,W_1} are independent of x, we have

$$\varphi_i(x) \leq C_1 \cdot C_2 + \frac{\ln C_{V,W_1}}{2} + 2 \qquad \text{for every } x \in V \cap f^{-1}(U_i).$$

The claim is proved. □

6. Proof of the main theorem

We start with an easy, but important lemma relating null locus and lc centres.

6.1. Lemma. *Let X be a compact Kähler manifold, and let α be a nef and big class such that the null locus Null(α) has no divisorial components. Let $Z \subset X$ be an irreducible component of Null(α). Then there exists a positive real number c such that Z is a maximal lc centre for $(X, c\alpha)$.*

Remark. The coefficient c depends on the choice of Z, so in general the other irreducible components of $\mathrm{Null}(\alpha)$ will not be lc centres for $(X, c\alpha)$.

Proof. By a theorem of Collins of Tosatti [CT13, Thm.1.1] the non-Kähler locus $E_{nK}(\alpha)$ coincides with the null-locus of $\mathrm{Null}(\alpha)$. Moreover by [Bou04, Thm.3.17] there exists a Kähler current T with analytic singularities in the class α such that the Lelong set coincides with $E_{nK}(\alpha)$. Since the non-Kähler locus has no divisorial components the class α is a modified Kähler class [Bou04, Defn.2.2]. By [Bou04, Prop.2.3] the class α has a log-resolution $\mu : \tilde{X} \to X$ such that $\mu_* \tilde{\alpha} = \alpha$. In fact the proof proceeds by desingularising a Kähler current with analytic singularities in the class α, so, using the current T defined above, we see that the μ-exceptional locus maps exactly onto $\mathrm{Null}(\alpha)$. Up to blowing up further the exceptional locus is a SNC divisor. By Remark 4.3 we have

$$\mu^* \alpha = \tilde{\alpha} + \sum_{j=1}^{k} r_j D_j.$$

with $r_j > 0$ for all $j \in \{1, \ldots, k\}$. Since α is nef and big, the class $\tilde{\alpha} + m\mu^*\alpha$ is Kähler for all $m > 0$. Thus up to replacing the decomposition above by

$$\mu^* \alpha = \frac{\tilde{\alpha} + m\mu^*\alpha}{m+1} + \sum_{j=1}^{k} \frac{r_j}{m+1} D_j$$

for $m \gg 0$ we can suppose that $r_j < 1$ for all $j \in \{1, \ldots, k\}$. Since X is smooth we have $K_{\tilde{X}} = \mu^* K_X + \sum_{j=1}^k a_j E_j$ with a_j a positive integer. Since $r_j < 1$ we have $a_j - r_j > -1$ for all E_j mapping onto Z. Thus we can choose a $c \in \mathbb{R}^+$ such that $a_j - cr_j \geq -1$ for all E_j mapping onto Z and equality holds for at least one divisor. □

As a first step toward Theorem 1.3 we can now prove the following:

6.2. Theorem. *Let X be a compact Kähler manifold of dimension n. Suppose that Conjecture 1.2 holds for all manifolds of dimension at most $n-1$. Suppose that K_X is pseudoeffective but not nef, and let ω be a Kähler class on X such that $\alpha := K_X + \omega$ is nef and big but not Kähler.*

Let $Z \subset X$ be an irreducible component of maximal dimension of the null-locus Null(α), and let $\pi : Z' \to Z$ be the composition of the normalisation and a resolution of singularities. Let k be the numerical dimension of $\pi^*\alpha|_Z$ (cf. Definition 2.5). Then we have

$$K_{Z'} \cdot \pi^*\alpha|_Z^k \cdot \pi^*\omega|_Z^{\dim Z-k-1} < 0.$$

In particular Z' is uniruled.

Proof of Theorem 6.2. Since $\alpha = K_X + \omega$ and $\pi^*\alpha|_Z^{k+1} = 0$ we have

$$\pi^*K_X|_Z \cdot \pi^*\alpha|_Z^k = -\pi^*\omega|_Z \cdot \pi^*\alpha|_Z^k.$$

By hypothesis $k < \dim Z$ so $\dim Z - k - 1$ is non-negative. Since $\pi^*\alpha|_Z^k$ is a non-zero nef class and ω is Kähler this implies by Remark 2.6 that

$$\pi^*K_X|_Z \cdot \pi^*\alpha|_Z^k \cdot \pi^*\omega|_Z^{\dim Z-k-1} = -\pi^*\omega|_Z^{\dim Z-k} \cdot \pi^*\alpha|_Z^k < 0. \tag{61}$$

Our goal will be to prove that

$$K_{Z'} \cdot \pi^*\alpha|_Z^k \cdot \pi^*\omega|_Z^{\dim Z-k-1} < 0.$$

This inequality implies the statement: since $K_{Z'}$ is not pseudoeffective and Conjecture 1.2 holds in dimension at most $n-1 \geq \dim Z'$ we obtain that Z' is uniruled.

We will make a case distinction:

Step 1. The null-locus of α contains an irreducible divisor. Since Z has maximal dimension, it is a divisor. Since K_X is pseudoeffective we can consider the divisorial Zariski decomposition [Bou04, Defn.3.7]

$$c_1(K_X) = \sum e_i Z_i + P(K_X),$$

where $e_i \geq 0$, the $Z_i \subset X$ are prime divisors and $P(K_X)$ is a modified nef class [Bou04, Defn.2.2]. Arguing as in [HP13, Lemma 4.1] we see that the inequality (61) implies (up to renumbering) that $Z_1 = Z$ and

$$\pi^*Z|_Z \cdot \pi^*\alpha|_Z^k \cdot \pi^*\omega|_Z^{n-k-2} < 0. \tag{62}$$

Moreover there exist effective $\mathbb{Q}$-divisors on D_1 and D_2 on Z' such that

$$K_{Z'} = \pi^*(K_X + Z) + D_1 - D_2$$

and $\pi(D_1)$ has codimension at least two in Z (cf. [Rei94, Prop.2.3]). Thus we have

$$K_{Z'} \cdot \pi^*\alpha|_Z^k \cdot \pi^*\omega|_Z^{n-k-2} \leq \pi^*(K_X + Z) \cdot \pi^*\alpha|_Z^k \cdot \pi^*\omega|_Z^{n-k-2}.$$

Combining (61) and (62) we obtain that the right hand side is negative.

Step 2. The null-locus of α has no divisorial components. In this case we know by Lemma 6.1 that there exists a $c > 0$ such that Z is a maximal lc centre for $(X, c\alpha)$. The classes $\pi^*\alpha|_Z$ and $\pi^*\omega|_Z$ are nef, so by Theorem 1.5 we have

$$K_{Z'} \cdot \pi^*\alpha|_Z^k \cdot \pi^*\omega|_Z^{\dim Z - k - 1} \leq \pi^*(K_X + c\alpha)|_Z \cdot \pi^*\alpha|_Z^k \cdot \pi^*\omega|_Z^{\dim Z - k - 1}.$$

Since k is the numerical dimension of $\pi^*\alpha|_Z$ we have $c\, \pi^*\alpha|_Z^{k+1} \cdot \pi^*\omega|_Z^{\dim Z - k - 1} = 0$. Thus (61) yields the claim. $\square$

6.3. Remark. We used the hypothesis that Z has maximal dimension only in Step 1, so our proof actually yields a more precise statement: $\mathrm{Null}(\alpha)$ contains a uniruled divisor or all the components of $\mathrm{Null}(\alpha)$ are uniruled.

We come now to the technical problem mentioned in the introduction:

6.4. Problem. *Let X be a compact Kähler manifold, and let $\alpha \in N^1(X)$ be a nef cohomology class. Does there exist a real number $b > 0$ such that for every (rational) curve $C \subset X$ we have either $\alpha \cdot C = 0$ or $\alpha \cdot C \geq b$?*

6.5. Remark. If α is the class of a nef $\mathbb{Q}$-divisor, the answer is obviously yes: some positive multiple $m\alpha$ is integral, so we can choose $b := \frac{1}{m}$. If α is a Kähler class the answer is also yes: by Bishop's theorem there are only finitely many deformation families of curves C such that $\alpha \cdot C \leq 1$, so $\alpha \cdot C$ takes only finitely many values in $]0, 1[$. However, even for the class of an $\mathbb{R}$-divisor on a projective manifold X it seems possible that the values $\alpha \cdot C$ accumulate at 0 [Laz04, Rem.1.3.12]. In the proof of Theorem 1.3 we will use that α is an adjoint class to obtain the existence of the lower bound b.

The problem 6.4 is invariant under certain birational morphisms:

6.6. Lemma. *Let $\pi : X \to X'$ be a holomorphic map between normal projective varieties X and X'. Let α' be a nef $\mathbb{R}$-divisor class on X' and set $\alpha := \pi^*\alpha'$.*

a) Suppose that there exists a real number $b > 0$ such that for every (rational) curve $C' \subset X'$ we have $\alpha' \cdot C' = 0$ or $\alpha' \cdot C' \geq b$. Then for every (rational) curve $C \subset X$ we have $\alpha \cdot C = 0$ or $\alpha \cdot C \geq b$.

b) Suppose that there exists a real number $b > 0$ such that for every (rational) curve $C \subset X$ we have $\alpha \cdot C = 0$ or $\alpha \cdot C \geq b$. Suppose also that X has klt singularities and π is the contraction of a K_X-negative extremal ray. Then for every (rational) curve $C' \subset X'$ we have $\alpha' \cdot C' = 0$ or $\alpha' \cdot C' \geq b$.

Proof. Proof of a) Let $C \subset X$ be a (rational) curve such that $\alpha \cdot C \neq 0$. the image $C' := \pi(C) \subset X'$ is a (rational) curve and the induced map $C \to C'$ has degree $d \geq 1$. Thus the projection formula yields

$$\alpha \cdot C = \pi^*\alpha' \cdot C = \alpha' \cdot \pi_*(C) = d\alpha' \cdot C' \geq db \geq b.$$

Proof of b) Let $C' \subset X'$ be an arbitrary (rational) curve such that $\alpha' \cdot C' \neq 0$. By [HM07, Cor.1.7(2)] the natural map $\pi^{-1}(C') \to C'$ has a section, so there exists a (rational) curve $C \subset X$ such that the map $\pi|_C : C \to C'$ has degree one. Thus the projection formula yields

$$\alpha' \cdot C' = \alpha' \cdot \pi_*(C) = \pi^*\alpha \cdot C \geq b.$$

$\square$

6.7. Remark. It is easy to see that statement a) also holds when X and X' are compact Kähler manifolds and α' is a nef cohomology class on X'.

6.8. Corollary. *Let X be a normal projective $\mathbb{Q}$-factorial variety with klt singularities, and let α be a nef $\mathbb{R}$-divisor class on X. Suppose that there exists a real number $b > 0$ such that for every (rational) curve $C \subset X$ we have $\alpha \cdot C = 0$ or $\alpha \cdot C \geq b$. Let $\mu : X \dashrightarrow X'$ be the divisorial contraction or flip of a K_X-negative extremal ray Γ such that $\alpha \cdot \Gamma = 0$. Set $\alpha' := \mu_*(\alpha)$. Then α' is a nef $\mathbb{R}$-divisor class on X' and for every (rational) curve $C \subset X$ we have $\alpha \cdot C = 0$ or $\alpha \cdot C \geq b$.*

Proof. If μ is divisorial the condition $\alpha \cdot \Gamma = 0$ implies that $\alpha = \mu^* \alpha'$ [KM98, Cor.3.17]. Thus Lemma 6.6, b) applies. If μ is a flip, let $f : X \to Y$ be the contraction of the extremal ray and $f' : X' \to Y$ the flipping map. Since $\alpha \cdot \Gamma = 0$ there exists an $\mathbb{R}$-divisor class α_Y on Y such that $\alpha = f^* \alpha_Y$ [KM98, Cor.3.17]. Moreover we have $\alpha' = (f')^* \alpha_Y$ since they coincide in the complement of the flipped locus. Thus we conclude by applying Lemma 6.6,b) to f and Lemma 6.6,a) to f'. □

6.9. Proposition. *Let F be a projective manifold, and let α be a nef $\mathbb{R}$-divisor class on F. Suppose that there exists a real number $b > 0$ such that for every rational curve $C \subset F$ such that $\alpha \cdot C \neq 0$ we have*

$$\alpha \cdot C > b. \tag{63}$$

Then one of the following holds

- *F is dominated by rational curves $C \subset F$ such that $\alpha \cdot C = 0$; or*
- *the class $K_F + \frac{2 \dim F}{b} \alpha$ is pseudoeffective.*

Proof. Note that, up to replacing α by $\frac{2 \dim F}{b} \alpha$, we can suppose that

$$\alpha \cdot C > 2 \dim F \tag{64}$$

for every rational curve $C \subset F$ that is not α-trivial. Suppose that $K_F + \alpha$ is not pseudoeffective, then our goal is to show that F is covered by α-trivial rational curves. Since $K_F + \alpha$ is not pseudoeffective, there exists an ample $\mathbb{R}$-divisor H such that $K_F + \alpha + H$ is not pseudoeffective. Since H and $\alpha + H$ are ample we can choose effective $\mathbb{R}$-divisors $\Delta_H \sim_{\mathbb{R}} H$ and $\Delta \sim_{\mathbb{R}} \alpha + H$ such that the pairs (F, Δ_H) and (F, Δ) are klt. By [BCHM10, Cor.1.3.3] we can run a $K_F + \Delta$-MMP

$$(F, \Delta) =: (F_0, \Delta_0) \overset{\mu_0}{\dashrightarrow} (F_1, \Delta_1) \overset{\mu_1}{\dashrightarrow} \ldots \overset{\mu_k}{\dashrightarrow} (F_k, \Delta_k),$$

that is for every $i \in \{0, \ldots, k-1\}$ the map $\mu_i : F_i \dashrightarrow F_{i+1}$ is either a divisorial Mori contraction of a $K_{F_i} + \Delta_i$-negative extremal ray Γ_i in $\overline{\mathrm{NE}}(X_i)$ or the flip of a small contraction of such an extremal ray. Note that for every $i \in \{0, \ldots, k\}$ the variety F_i is normal $\mathbb{Q}$-factorial and the pair (F_i, Δ_i) is klt. Moreover F_k admits a Mori contraction of fibre type $\psi : F_k \to Y$ contracting an extremal ray Γ_k such that $(K_{F_k} + \Delta_k) \cdot \Gamma_k < 0$.

Set $\Delta_{H,0} := \Delta_H, \alpha_0 := \alpha$ and for all $i \in \{0, \ldots, k-1\}$ we define inductively

$$\Delta_{H,i+1} := (\mu_i)_*(\Delta_{H,i}), \ \alpha_{i+1} := (\mu_i)_*(\alpha_i).$$

Note that for all $i \in \{0, \ldots, k\}$ we have

$$K_{F_i} + \Delta_i \equiv K_{F_i} + \Delta_{H,i} + \alpha_i. \tag{65}$$

We claim that for all $i \in \{0, \dots, k\}$ the $\mathbb{R}$-divisor class α_i is nef and $\alpha_i \cdot \Gamma_i = 0$. Moreover the pairs $(X_i, \Delta_{H,i})$ are klt. Assuming this for the time being, let us see how to conclude: since $\psi : F_k \to Y$ is a Mori fibre space and the extremal ray Γ_k is α_k-trivial, we see that F_k is dominated by α_k-trivial rational curves $(C_t)_{t \in T}$. A general member of this family of rational curves is not contained in the exceptional locus of $F_0 \dashrightarrow F_k$, so the strict transforms define a dominant family of rational curves $(C'_t)_{t \in T}$ of F_0. Since all the birational contractions in the MMP $F_0 \dashrightarrow F_k$ are $\alpha_\bullet$-trivial, we easily see (cf. the proof of Corollary 6.8) that

$$\alpha \cdot C'_t = \alpha_k \cdot C_t = 0.$$

Proof of the claim. Since α_0 is nef, we have

$$0 > (K_{F_0} + \Delta_0) \cdot \Gamma_0 = (K_{F_0} + \Delta_{H,0} + \alpha_0) \cdot \Gamma_0 \geq (K_{F_0} + \Delta_{H,0}) \cdot \Gamma_0.$$

Thus the extremal ray Γ_0 is $K_{F_0} + \Delta_{H,0}$-negative, in particular the pair (F_1, Δ_1) is klt [KM98, Cor.3.42, 3.43]. Moreover there exists by [Kaw91, Thm.1] a rational curve $[C_0] \in \Gamma_0$ such that $(K_{F_0} + \Delta_{H,0}) \cdot C_0 \geq -2 \dim F$. Thus if $\alpha_0 \cdot C_0 \neq 0$, the inequality (64) implies that

$$(K_{F_0} + \Delta_0) \cdot C_0 = (K_{F_0} + \Delta_{H,0}) \cdot C_0 + \alpha_0 \cdot C_0 > 0.$$

In particular the extremal ray Γ_0 is not $K_{F_0} + \Delta_0$-negative, a contradiction to our assumption. Thus we have $\alpha_0 \cdot C_0 = 0$. By Corollary 6.8 this implies that α_1 is nef and satisfies the inequality (64). The claim now follows by induction on i. □

6.10. Remark. For the proof of Theorem 1.3 we will use the MRC fibration of a uniruled manifold. Since the original papers [KMM92, Cam92] are formulated for projective manifolds, let us recall that for a compact Kähler manifold M that is uniruled the MRC fibration is defined as an almost holomorphic map $f : M \dashrightarrow N$ such that the general fibre F is rationally connected and the dimension of F is maximal among all the fibrations of this type. The existence of the MRC fibration follows, as in the projective case, from the existence of a quotient map for covering families [Cam04]. The base N is not uniruled : arguing by contradiction we consider a dominating family $(C_t)_{t \in T}$ of rational curves on N. Let M_t be a desingularisation of $f^{-1}(C_t)$ for a general C_t, then M_t is a compact Kähler manifold with a fibration onto a curve $M_t \to C_t$ such that the general fibre is rationally connected. In particular $H^0(M_t, \Omega^2_{M_t}) = 0$ so M_t is projective by Kodaira's criterion. Thus we can apply the Graber-Harris-Starr theorem [GHS03] to see that M_t is rationally connected, a contradiction.

Proof of Theorem 1.3. Let ω be a Kähler class such that $\alpha := K_X + \omega$ is nef and big, but not Kähler. By Theorem 6.2 there exists a subvariety $Z \subset X$ contained in the null-locus $\mathrm{Null}(\alpha)$ that is uniruled. More precisely let $\pi : Z' \to Z$ be a desingularisation, and denote by k the numerical dimension of $\alpha' := \pi^*\alpha|_Z$. Then we know by Theorem 6.2 that

$$K_{Z'} \cdot \alpha'^k \cdot \pi^*\omega|_Z^{\dim Z - k - 1} < 0.$$

Since $\alpha'^{k+1} = 0$ this actually implies that

$$(66) \qquad (K_{Z'} + \lambda\alpha') \cdot \alpha'^k \cdot \pi^*\omega|_Z^{\dim Z - k - 1} < 0 \qquad \forall\, \lambda > 0.$$

Our goal is to prove that this implies that Z contains a K_X-negative rational curve. Arguing by contradiction we suppose that $K_X \cdot C \geq 0$ for every rational curve $C \subset Z$. Since ω is a Kähler class this implies by Remark 6.5 that there exists a $b > 0$ such that for every rational curve $C \subset Z$ we have

$$\alpha \cdot C = (K_X + \omega) \cdot C \geq \omega \cdot C \geq b. \tag{67}$$

By Lemma 6.6a) and Remark 6.7 this implies that for every rational curve $C' \subset Z'$ we have $\alpha' \cdot C' = 0$ or $\alpha' \cdot C' \geq b$.

Since Z' is uniruled we can consider the MRC-fibration $f : Z' \dashrightarrow Y$ (cf. Remark 6.10). The general fibre F is rationally connected, in particular we can consider $\alpha'|_F$ as a nef $\mathbb{R}$-divisor class. Moreover the inequality above shows that $\alpha'|_F$ satisfies the condition (63) in Proposition 6.9. If F is dominated by $\alpha'|_F$-trivial rational curves, then Z' is dominated by α'-trivial rational curves. A general member of this dominating family is not contracted by π, so Z is dominated by α-trivial rational curves. This possibility is excluded by (67), so Proposition 6.9 shows that there exists a $\lambda > 0$ such that $K_F + \lambda\alpha'|_F$ is pseudoeffective.

We will now prove that $K_{Z'} + \lambda\alpha$ is pseudoeffective, which clearly contradicts (66). If $\nu : Z'' \to Z$ is a resolution of the indeterminacies of f such that $K_{Z''} + \nu^*(\lambda\alpha)$ is pseudoeffective, then $K_{Z'} + \lambda\alpha = (\nu)_*(K_{Z''} + \nu^*(\lambda\alpha))$ is pseudoeffective. Thus we can assume without loss of generality that the MRC-fibration f is a holomorphic map. Let ω' be a Kähler class on Z', then for every $\varepsilon > 0$ the class $\lambda\alpha' + \varepsilon\omega$ is Kähler and $K_F + (\lambda\alpha + \varepsilon\omega)|_F$ is pseudoeffective. Thus we can apply Theorem 5.4 to $f : Z' \to Y$ to see that

$$K_{Z'/Y} + \lambda\alpha + \varepsilon\omega$$

is pseudoeffective. Note now that Y has dimension at most $\dim X - 2$ and is not uniruled (Remark 6.10) Since we assume that Conjecture 1.2 holds in dimension up to $\dim X - 1$, we obtain that K_Y is pseudoeffective. Thus we see that $K_{Z'} + \lambda\alpha + \varepsilon\omega$ is pseudoeffective for all $\varepsilon > 0$. The statement follows by taking the limit $\varepsilon \to 0$. □

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